MATHEMATICA, Tome 46 (69), N° 1, 2004, pp. 55–60

QUASICONFORMAL EXTENSIONS OF HOLOMORPHIC MAPS IN $\mathbb{C}^n$

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Abstract. Let B be the unit ball in $\mathbb{C}^n$ with respect to the euclidian norm. In this paper we will give a sufficient condition such that a holomorphic mapping defined on B can be extended to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

MSC 2000. 30C65.
Key words. Holomorphic maps, subordination.

1. INTRODUCTION AND PRELIMINARIES

J. Becker [1] showed that if f is a holomorphic function on the unit disc U which satisfies

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{q}{1-|z|^2} \quad (0<q<1)\,,$$

then f is univalent on U and extends to a quasiconformal homeomorphism of $\mathbb{R}^2$ onto itself.

This result was generalized by J.A. Pfaltzgraff [9] to several complex variables. He showed that if f is a quasiregular holomorphic mapping defined on B and satisfies the following condition

$$(1-\|z\|^2)\|[Df(z)]^{-1}D^2f(z)(z,\cdot)\| \leq q \quad (0<q<1)$$

($\|\cdot\|$ denotes the euclidian norm on $\mathbb{C}^n$), then f is biholomorphic on B and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Recently, the problem of quasiconformal extensions for holomorphic mappings was also studied by M. Chuaqui [3], H. Hamada and G. Kohr [7], [8], P. Curt [5].

In this paper we shall generalize the result due to J.A. Pfatzgraff [9] (mentioned in the previous paragraph).

Let $\mathbb{C}^n$ denote the space of n complex variables $z=(z_1,\dots,z_n)'$ with the usual inner product $\langle z,w\rangle = \sum_{i=1}^{n} z_i\overline{w}_i$ and euclidian norm $\|z\| = \sqrt{\langle z,z\rangle}$. The symbol $'$ means the transpose of vectors. Let B denote the open unit ball in $\mathbb{C}^n$. We denote by $\mathcal{L}(\mathbb{C}^n)$ the space of linear operators from $\mathbb{C}^n$ into $\mathbb{C}^n$, i.e. the $n\times n$ complex matrices $A=(A_{jk})$, with the standard operator norm.

The class of holomorphic mappings $f(z)=(f_1(z),\dots,f_n(z))'$, $z\in B$, from B into $\mathbb{C}^n$ is denoted by $\mathcal{H}(B)$. We say that $f\in\mathcal{H}(B)$ is locally biholomorphic (locally univalent) in B if f has a local holomorphic inverse at each point in

B, or equivalently, if the derivative

$$Df(z) = \left(\frac{\partial f_k(z)}{\partial z_j} \right)_{j,k}$$

is nonsingular at each point $z \in B$. If $f \in \mathcal{H}(B)$, we say that f is biholomorphic on B if the inverse f^{-1} exists and is holomorphic on the domain $f(B)$.

If $f, g \in \mathcal{H}(B)$, we say that f is subordinate to g if there exists a Schwarz function v such that $f(z) = g(v(z))$, $z \in B$. We shall write $f \prec g$ to mean that f is subordinate to g.

We say that $f \in \mathcal{H}(B)$ is quasiregular in B if there exists a constant $K > 0$ such that

$$\|Df(z)\|^n \leq K |\det Df(z)|, \quad z \in B. \tag{1.1}$$

It is known that a holomorphic and quasiregular mapping is locally biholomorphic [2].

Let G and G' be domain in $\mathbb{R}^m$. Let $|\cdot|$ be an arbitrary norm on $\mathbb{R}^m$. We say that a homeomorphism $f : G \to G'$ is k-quasiconformal if it is differentiable a.e., $A \subset L$ (absolutely continuous on lines) and

$$|Df(x)|^m \leq K |\det Df(x)| \text{ a.e. in } G$$

where $Df(x)$ denotes the (real) Jacobian matrix of f at x. The definition of quasiconformality is independent of the choice of a norm on $\mathbb{R}^m$.

Definition 1.1. The mapping $L : B \times [0, \infty) \to \mathbb{C}^n$ is called a normalized Loewner chain (normalized subordination chain) if

(i) $L(\cdot, t)$ is biholomorphic on B, $t \geq 0$

(ii) $L(0, t) = 0$, $DL(0, t) = \mathrm{e}^t I$, $t \geq 0$

(iii) $L(\cdot, s) \prec L(\cdot, t)$ for $0 \leq s \leq t < \infty$

(iv) $L(z, \cdot)$ is a locally absolutely continuous function of $t \in [0, \infty)$ locally uniformly with respect to $z \in B$.

Note that in the case of one variable, the assumption (iv) is always satisfied as a consequence of the distorsion result for the class of normalized univalent functions defined on the disc U.

An important role in our discussion is played by the n-dimensional version of the Caratheodory set:

$$\mathcal{M} = \{h \in \mathcal{H}(B) : \ h(0) = 0, \ Dh(0) = I, \ \mathrm{Re}\, \langle h(z), z \rangle \geq 0, \ z \in B\}.$$

Recently, in [6] P. Curt and G. Kohr proved that normalized subordination chains satisfy the generalized Loewner differential equation.

Theorem 1.1. *Let $L : B \times [0, \infty) \to \mathbb{C}^n$ be a normalized subordination chain. Then there exists $E \subset (0, \infty)$ a set of Lebesgue measure zero such that* *for every* $t \in (0,\infty) \setminus E$ *there exists* $h(z,t)$ *such that* $h(\cdot,t) \in \mathcal{M}$, $h(z,\cdot)$ *is measurable on* $[0,\infty)$ *for each* $z \in B$ *and*

$$\frac{\partial L}{\partial t}(z,t) = DL(z,t)h(z,t), \quad t \in (0,\infty) \setminus E,\ z \in B. \tag{1.2}$$

Definition 1.2. [5] Let $L : B\times[0,\infty) \to \mathbb{C}^n$ be a normalized subordination chain and let $q \in [0,1)$.

We say that L is a q-normalized subordination chain if the mapping h defined by Theorem 1.1 satisfies the following conditions:

(i)

$$\|z\|^2\frac{1-q\|z\|}{1+q\|z\|} \le \mathrm{Re}\ \langle h(z,t), z\rangle \le \|z\|^2\frac{1+q\|z\|}{1-q\|z\|}, \tag{1.3}$$

$z \in B$, a.e. $t \in [0,\infty)$

(ii) there is a constant $q_1 > 0$ such that

$$\|h(z,t)\| < q_1, \quad z \in B,\ \text{a.e. } t \in [0,\infty). \tag{1.4}$$

In the next remark, we will present a class of holomorphic mappings which satisfy the conditions (1.3) and (1.4).

Remark 1.1. [5] Let $q \in [0,1)$ and $h : B \times [0,\infty) \to \mathbb{C}^n$ defined by

$$h(z,t) = [I - E(z,t)]^{-1}[I + E(z,t)](z)\,,$$

where E satisfies the following conditions:

(i) $E(z,t) \in \mathcal{L}(\mathbb{C}^n),\ z \in B,\ t \in [0,\infty)$

(ii) $E(\cdot,t) : B \to \mathcal{L}(\mathbb{C}^n)$ is a holomorphic mapping

(iii) $E(0,t) = 0,\ \|E(z,t)\| \le q$.

Then h satisfies the conditions (1.3) and (1.4).

We shall need the following theorem to prove our results.

Theorem 1.2. [5] *Let* $q \in [0,1)$ *and* $L : B\times[0,\infty) \to \mathbb{C}^n$ *be a q-normalized subordination chain. Assume that the following conditions are satisfied:*

(i) $L(\cdot,t)$ *is quasiregular for each* $t \in [0,\infty)$

(ii)

$$\|DL(z,t)\| \le \frac{\mathrm{e}^t M}{(1-\|z\|)^\alpha}, \quad z \in B,\ t \in [0,\infty)\,, \tag{1.5}$$

where $M > 0$ *and* $0 \le \alpha < 1$

(iii) *there is a sequence* $\{t_m\}_m$, $t_m > 0$, *increasing to* ∞ *and a mapping* $F \in \mathcal{H}(B)$ *such that*

$$\lim_{m\to\infty}\frac{L(z,t_m)}{\mathrm{e}^{t_m}} = F(z),\ \text{locally uniformly in } B. \tag{1.6}$$

Then $f(z) = L(z,0)$ *admits a quasiconformal extension to* $\mathbb{R}^{2n}$.

2. MAIN RESULTS

THEOREM 2.1. *Let $f, g \in \mathcal{H}(B)$ such that $f(0) = g(0) = 0$, $Df(0) = Dg(0) = I$ and g is quasiregular in B. If there is $q \in [0,1)$ such that*

$$\|[Dg(z)]^{-1}Df(z) - I\| < q, \quad z \in B \tag{2.1}$$

and

$$\left\| \|z\|^2 \{[Dg(z)]^{-1}Df(z) - I\} + (1 - \|z\|^2)[Dg(z)]^{-1}D^2g(z)(z, \cdot) \right\| < q, \quad z \in B, \tag{2.2}$$

then f extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Proof. We shall show that the conditions (2.1) and (2.2) enable us to imbed f as the initial element $f(z) = L(z, 0)$ of a suitable normalized chain.

We define

$$L(z,t) = f(\mathrm{e}^{-t}z) + (\mathrm{e}^{t} - \mathrm{e}^{-t})Dg(z\mathrm{e}^{-t})(z), \quad (z,t) \in B \times [0, \infty). \tag{2.3}$$

In [4] the authors proved that the mapping L defined by (2.3) is a normalized subordination chain. In the same paper the authors showed that the subordination chain defined by (2.3) satisfies the generalized Loewner equation (1.2) where the mapping h is defined by

$$h(z,t) = [I - E(z,t)]^{-1}[I + E(z,t)](z), \quad (z,t) \in [0, \infty) \tag{2.4}$$

and the mapping $E : B \times [0, \infty) \to \mathcal{L}(\mathbb{C}^n)$ is defined by

$$\begin{aligned} E(z,t) &= \mathrm{e}^{-2t}\{[Dg(\mathrm{e}^{-t}z)]^{-1}Df(\mathrm{e}^{-t}z) - I\} - \\ &- (1 - e^{-zt})[Dg(\mathrm{e}^{-t}z)]^{-1}D^2g(\mathrm{e}^{-t}z)(\mathrm{e}^{-t}z, \cdot). \end{aligned} \tag{2.5}$$

Further, we shall show that $\|E(z,t)\| \le q$ for all $(z,t) \in B \times [0, \infty)$. We consider:

$$A(\mathrm{e}^{-t}z) = [Dg(\mathrm{e}^{-t}z)]^{-1}Df(\mathrm{e}^{-t}z) - I\,,$$

$$B(\mathrm{e}^{-t}z) = [Dg(\mathrm{e}^{-t}z)]^{-1}D^2g(\mathrm{e}^{-t}z)(\mathrm{e}^{-t}z, \cdot)$$

and

$$F(z,t,\lambda) = \lambda A(\mathrm{e}^{-t}z) + (1-\lambda)B(\mathrm{e}^{-t}z), \quad \lambda \in [0,1].$$

From (2.1) and (2.2) it results $\|A(\mathrm{e}^{-t}z)\| \le q$ and $\|F(z,t,\lambda_z)\| \le q$ where $\lambda_z = \mathrm{e}^{-2t}\|z\|^2$, $z \in B$, $t \ge 0$. Since $1 \ge \mathrm{e}^{-2t} > \lambda_z$, for all $z \in B$ and $t \ge 0$ we can write $\mathrm{e}^{-2t} = u + (1-u)\lambda_z$, where $u \in [0,1)$. Then

$$-E(z,t) = uA(\mathrm{e}^{-t}z) + (1-u)F(z,t,\lambda_z), \quad u \in [0,1).$$

We obtain

$$\|E(z,t)\| \le u\|A(\mathrm{e}^{-t}z)\| + (1-u)\|F(z,t,\lambda_z)\| \le q, \quad (z,t) \in B \times [0, \infty)$$

and hence $I - E(z,t)$ is an invertible operator.

Further calculation shows that

$$\frac{\partial L(z,t)}{\partial t} = e^t Dg(e^{-t}z)[I + E(z,t)](z) \tag{2.6}$$

$$= DL(z,t)[I - E(z,t)]^{-1}[I + E(z,t)](z). \tag{2.7}$$

It results that $L(z,t)$ satisfies the differential equation (1.2) for all $t \geq 0$ and $z \in B$, where

$$h(z,t) = [I - E(z,t)]^{-1}[I + E(z,t)](z).$$

Hence L is a q-normalized subordination chain which satisfies (1.6) [4].

Next, we will prove (1.5). By using the fact that g is a quasiregular mapping which satisfies (2.1) and (2.2) we will show that the mapping g satisfies the condition (1.5). By using (2.1) and (2.2) we obtain first

$$(1 - \|z\|^2)\|[Dg(z)]^{-1}D^2g(z)(z,\cdot)\| \leq 2q, \quad z \in B. \tag{2.8}$$

From (2.6), by using a similar argument with that used in the proof of Theorem 2.1 [9] we obtain that there exists $M > 0$ such that

$$\|Dg(z)\| \leq \frac{M}{(1 - \|z\|)^q}, \quad z \in B. \tag{2.9}$$

The equality

$$e^{-t}DL(z,t) = Dg(ze^{-t})[I - E(z,t)]$$

implies that

$$\|e^{-t}DL(z,t)\| \leq \|Dg(ze^{-t})\| \cdot \|I - E(z,t)\| \tag{2.10}$$

$$\leq \frac{M(1+q)}{(1 - \|ze^{-t}\|)^q} \leq \frac{M(1+q)}{(1 - \|z\|)^q}, \quad z \in B,\ t \geq 0. \tag{2.11}$$

It remains to prove that the mappings $L(\cdot,t)$, $t \geq 0$, are quasiregular. For the subordination chain defined by (2.3) we have

$$DL(z,t) = e^t Dg(e^{-t}z)[I - E(z,t)], \quad z \in B,\ t \geq 0.$$

Since g is a quasiregular holomorphic mapping and the following inequality holds

$$1 - q \leq \|I - E(z,t)\| \leq 1 + q, \quad z \in B,\ t \geq 0$$

we easily obtain

$$\begin{aligned} \|DL(z,t)\|^n &\leq e^{nt}\|Dg(e^{-t}z)\|^n\|I - E(z,t)\|^n \\ &\leq e^{nt}(1+q)^n K|\det Dg(e^{-t}z)| \\ &= \frac{(1+q)^n K}{|\det[I - E(z,t)]|}|\det DL(z,t)| \\ &\leq \left(\frac{1+q}{1-q}\right)^n K|\det DL(z,t)|, \quad z \in B,\ t \geq 0. \end{aligned} \tag{2.12}$$

Since the conditions of Theorem 1.2 are satisfied we obtain that the function $f(z) = L(z,\cdot)$ admits a quasiconformal extension defined on $\mathbb{R}^{2n}$. □

Observe that if $f = g$, we obtain Theorem 3.1 of [9].

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Received February 10, 2002

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