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GENERALIZED *RD*-PURE-INJECTIVITY AND *RD*-FLATNESS

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Abstract. Let R be an associative ring with non-zero identity and let V be a non-empty subset of R. We shall consider a family Ω_0 of left R-modules of the form R/Rr, where $r \in V$. If R is commutative, we shall determine the structure of Ω_0-pure-injective R-modules. We shall also study Ω_0-flat modules.

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1. INTRODUCTION

In this paper we denote by R an associative ring with non-zero identity and all R-modules are unital. By Mod-R we denote the category of right R-modules. By a homomorphism we understand an R-homomorphism. For the sake of brevity, we shall omit the writing of the homomorphisms induced by the functors Hom_R and tensor product.

Let Ω be a class of left R-modules and let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{1}$$

be a short exact sequence of right R-modules, where f and g are homomorphisms. If the tensor product $f \otimes_R 1_D : A \otimes_R D \to B \otimes_R D$ is a monomorphism for every $D \in \Omega$, it is said that the sequence (1) is Ω-pure [4]. If A is a submodule of B, f is the inclusion monomorphism and the sequence (1) is Ω-pure, then A is said to be an Ω-pure submodule of B.

A right R-module M is called projective with respect to the sequence (1) (or with respect to the epimorphism g) if the natural homomorphism $Hom_R(M, B) \to Hom_R(M, C)$ is surjective. A right R-module is called injective with respect to the sequence (1) (or with respect to the monomorphism f) if the natural homomorphism $Hom_R(B, M) \to Hom_R(A, M)$ is surjective. A right R-module E is said to be Ω-pure-injective if E is injective with respect to every Ω-pure short exact sequence of right R-modules.

A right R-module C is called Ω-flat if every short exact sequence (1) is Ω-pure [4].

The injective hull of an R-module A is denoted by $E(A)$. We denote by $(M_i)_{i \in I}$ the family of all distinct right ideals of R and $S_i = R/M_i$ for every $i \in I$.

If R is a commutative ring, K is an ideal of R, A is an R-module and $r \in R$, then we denote $Ann_A K = \{a \in A \mid ra = 0\,, \forall r \in K\}$ and $Ann_A r = Ann_A(Rr)$.

Let $V \subseteq R$ be a non-empty set. In this paper we shall consider the family of left R-modules

$$\Omega_0 = \{R/Rr \mid r \in V\}.$$

If $V = R$, then an Ω_0-pure exact sequence (1) is called RD-pure [5]. Notice that if the exact sequence (1) is RD-pure, then it is Ω_0-pure.

In [2] we characterized Ω_0-pure short exact sequences and we determined the structure of Ω_0-pure-projective modules.

In the present paper, for a commutative ring R, we shall determine the structure of Ω_0-pure-injective R-modules. We shall also characterize Ω_0-flat modules and study the class of Ω_0-flat modules.

2. Ω_0-PURITY AND Ω_0-PURE-INJECTIVITY

We shall recall three results which will be used later in the paper.

THEOREM 2.1. [4, Proposition 2.3] *Let T be a set of right R-modules which contains a family of cogenerators for Mod-R and let $p^{-1}(T)$ be the class of all short exact sequences in Mod-R with the property that every R-module in T is injective with respect to them. Then:*

(i) *For every right R-module L there exists a short exact sequence*

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

in $p^{-1}(T)$ with $M \in T$.

(ii) *Every right R-module which is injective with respect to each sequence in $p^{-1}(T)$ is a direct summand of a direct product of R-modules in T.*

LEMMA 2.2. [6, Lemma 7.16] *Consider the commutative diagram with exact rows in Mod-R*

$$\begin{array}{ccccccc}
 & & M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \\
0 & \longrightarrow & N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 & &
\end{array}$$

The following statements are equivalent:

(i) *There exists $\alpha : M_3 \to N_2$ with $g_2\alpha = \varphi_3$;*

(ii) *There exists $\beta : M_2 \to N_1$ with $\beta f_1 = \varphi_1$.*

THEOREM 2.3. [1, Theorem 7] *Let R be commutative, let A be an R-module and let K be an ideal of R. Then there exists an isomorphism of R-modules $\alpha : Hom_R(R/K, A) \to Ann_A K$, that is defined by $\alpha(f) = f(u)$, where $f \in Hom_R(R/K, A)$ and R/K is generated by u.*

The proof of the following result is the same as for the case of injective R-modules [3, Proposition 2.2].

LEMMA 2.4. *Let $(D_j)_{j\in J}$ be a family of right R-modules. Then the direct product $\prod_{j\in J} D_j$ is Ω_0-pure-injective if and only if D_j is Ω_0-pure-injective for every $j \in J$.*

Till the end of the present section, the ring R is assumed to be commutative.

THEOREM 2.5. *Let G be an injective R-module. Then $Hom_R(R/Rr, G)$ is an Ω_0-pure-injective module for every $r \in V$.*

Proof. Suppose that the exact sequence (1) is Ω_0-pure and let $r \in V$. Then the sequence

$$0 \longrightarrow A \otimes_R R/Rr \longrightarrow B \otimes_R R/Rr \longrightarrow C \otimes_R R/Rr \longrightarrow 0 \tag{2}$$

of R-modules is exact. Since G is injective, the sequence

$$\begin{aligned} 0 \longrightarrow Hom_R(C \otimes_R R/Rr, G) \longrightarrow Hom_R(B \otimes_R R/Rr, G) \\ \longrightarrow Hom_R(A \otimes_R R/Rr, G) \longrightarrow 0 \end{aligned} \tag{3}$$

of R-modules is exact. Using the isomorphism

$$Hom_R(N \otimes_R M, D) \cong Hom_R(N, Hom_R(M, D)),$$

where M, N, D are R-modules, we obtain the exact sequence:

$$\begin{aligned} 0 \longrightarrow Hom_R(C, Hom_R(R/Rr, G)) \longrightarrow Hom_R(B, Hom_R(R/Rr, G)) \\ \longrightarrow Hom_R(A, Hom_R(R/Rr, G)) \longrightarrow 0 \end{aligned} \tag{4}$$

Hence $Hom_R(R/Rr, G)$ is Ω_0-pure-injective. □

THEOREM 2.6. *Let G be a cogenerator for Mod-R and let $Hom_R(R/Rr, G)$ be injective with respect to the exact sequence* (1) *for every $r \in V$. Then the exact sequence* (1) *is Ω_0-pure.*

Proof. Let $r \in V$. Since $Hom_R(R/Rr, G)$ is injective with respect to the exact sequence (1), we obtain the exact sequence (4). Hence the sequence (3) is exact. But then the sequence (2) is exact, because G is a cogenerator. It follows that the sequence (1) is Ω_0-pure. □

COROLLARY 2.7. *Let G be an injective cogenerator for Mod-R. Then the exact sequence* (1) *is Ω_0-pure if and only if $Hom_R(R/Rr, G)$ is injective with respect to the exact sequence* (1) *for every $r \in V$.*

COROLLARY 2.8. *The exact sequence* (1) *is Ω_0-pure if and only if the R-module $Hom_R(R/Rr, E(S))$ is injective with respect to the exact sequence* (1) *for every $r \in V$ and for every simple R-module S.*

Proof. It is known that $\prod_{i\in I} E(S_i)$ is an injective cogenerator for Mod-R. Since S is a simple R-module, there exists $i \in I$ such that $S \cong S_i$. For every $r \in V$ we have the isomorphism:

$$Hom_R(R/Rr, \prod_{i\in I} E(S_i)) \cong \prod_{i\in I} Hom_R(R/Rr, E(S_i)).$$

Now the result follows by Lemma 2.4 and Corollary 2.7 if we take $G = \prod_{i\in I} E(S_i)$. $\square$

THEOREM 2.9. *The exact sequence* (1) *is* Ω_0*-pure if and only if* $Ann_{E(S)}r$ *is injective with respect to the exact sequence* (1) *for every simple* R*-module* S *and every* $r \in V \cap M$*, where* M *is a maximal ideal of* R *such that* $S \cong R/M$.

Proof. Let S be a simple R-module such that $S \cong R/M$, where M is a maximal ideal of R. Let $r \in V$. By Theorem 2.3, we have an isomorphism

$$Hom_R(R/Rr, E(S)) \cong Ann_{E(S)}r\,.$$

Let $r \notin M$. Suppose that $Ann_{E(S)}r \neq 0$. Then there exists $0 \neq a \in Ann_{E(S)}r$. Hence $ra = 0$. Since $a \in E(S)$, there exists $t \in R$ such that $0 \neq ta \in S$. Then $r(ta) = 0$. Since S is simple, it is generated by ta, hence $Ann_{R/M}r = R/M$. It follows that $r(1 + M) = M$, i.e., $r \in M$, which is a contradiction. Therefore, $Ann_{E(S)}r = 0$, which is an injective R-module.

Now let $r \in M$. Then $S \subseteq Ann_{E(S)}r$, hence $Ann_{E(S)}r \neq 0$.

The result follows by Corollary 2.8. $\square$

Now, by Theorems 2.1 and 2.9 we obtain the following two corollaries.

COROLLARY 2.10. *For every* R*-module* A*, there exists an* Ω_0*-pure short exact sequence of* R*-modules*

$$0 \longrightarrow A \longrightarrow M \longrightarrow N \longrightarrow 0$$

where M *is* Ω_0*-pure-injective.*

We are also able to establish the structure of Ω_0-pure-injective modules.

COROLLARY 2.11. *Every* Ω_0*-pure-injective* R*-module is a direct summand of a direct product of* R*-modules of the form* $Ann_{E(S)}r$*, where* S *is a simple* R*-module and* $r \in V \cap M$ *for some maximal ideal* M *of* R *such that* $S \cong R/M$.

3. Ω_0-FLAT MODULES

We begin the section with a technical result that will be useful for the characterization of Ω_0-flat modules.

LEMMA 3.1. *Consider the short exact sequence* (1)*. Let* E *be a right* R*-module which is injective with respect to* f *and let* $h : E \to D$ *be an epimorphism of right* R*-modules. If* A *and* C *are projective with respect to* h*, then* B *is projective with respect to* h.

Proof. Let $u : B \to D$ be a homomorphism. Since A is projective with respect to h, there exists a homomorphism $v : A \to E$ such that $uf = hv$.

Thus we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle v} & & \Big\downarrow{\scriptstyle u} & & & & \\ & & E & \xrightarrow{h} & D & \longrightarrow & 0 & & \end{array}$$

Since E is injective with respect to f, there exists a homomorphism $w : B \to E$ such that $wf = v$. Then $hwf = hv = uf$ and furthermore $(hw - u)f = 0$. Hence there exists a homomorphism $\alpha : C \to D$ such that $\alpha g = hw - u$. But C is projective with respect to h, so that there exists a homomorphism $\beta : C \to E$ such that $h\beta = \alpha$. Now consider the homomorphism $\gamma : B \to E$ defined by $\gamma = w - \beta g$. We have $h\gamma = hw - h\beta g = hw - \alpha g = u$. Therefore, B is projective with respect to h. □

In what follows, the ring R is assumed to be commutative.

THEOREM 3.2. *The following statements are equivalent:*
(i) *The exact sequence* (1) *is* Ω_0*-pure.*
(ii) *For every commutative diagram of* R*-modules*

$$\begin{array}{ccc} B & \xrightarrow{\quad g \quad} & C \\ \Big\downarrow{\scriptstyle u} & & \Big\downarrow{\scriptstyle v} \\ E(S) & \xrightarrow{h} & E(S)/Ann_{E(S)}r \end{array} \qquad (5)$$

where $S \cong R/M$ *for some maximal ideal* M *of* R*,* $r \in V \cap M$*,* h *is the natural projection and* u, v *are homomorphisms, there exists a homomorphism* $w : C \to E(S)$ *such that* $hw = v$*.*

Proof. (i) $\Longrightarrow$ (ii) Suppose that the exact sequence (1) is Ω_0-pure and consider the diagram (5). Let $k : Ann_{E(S)}r \to E(S)$ be the inclusion homomorphism. Then there exists a homomorphism $\alpha : A \to Ann_{E(S)}r$ such that the following diagram with exact rows is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle u} & & \Big\downarrow{\scriptstyle v} & & \\ 0 & \longrightarrow & Ann_{E(S)}r & \xrightarrow{k} & E(S) & \xrightarrow{h} & E(S)/Ann_{E(S)}r & \longrightarrow & 0 \end{array} \qquad (6)$$

By Theorem 2.9 $Ann_{E(S)}r$ is Ω_0-pure-injective, hence there exists a homomorphism $\beta : B \to Ann_{E(S)}r$ such that $\beta f = \alpha$. Now by Lemma 2.2, there exists a homomorphism $w : C \to E(S)$ such that $hw = v$.

(ii) $\Longrightarrow$ (i) Suppose that (ii) holds. Let $S \cong R/M$ for some maximal ideal M of R, let $r \in V \cap M$ and let $\alpha : A \to Ann_{E(S)}r$ be a homomorphism. By the injectivity of $E(S)$, there exists a homomorphism $u : B \to E(S)$ such that $uf = k\alpha$, where $k : Ann_{E(S)}r \to E(S)$ is the inclusion homomorphism. Then we can construct a commutative diagram of R-modules with exact rows (6), where $v : C \to E(S)/Ann_{E(S)}r$ is a homomorphism. Hence there exists a homomorphism $w : C \to E(S)$ such that $hw = v$. By Lemma 2.2, there exists a homomorphism $\beta : B \to Ann_{E(S)}r$ such that $\beta f = \alpha$. Now by Theorem 2.9, the exact sequence (1) is Ω_0-pure. □

THEOREM 3.3. *Let C be an R-module. The following statements are equivalent:*

(i) *C is Ω_0-flat.*

(ii) *For every simple R-module $S \cong R/M$, where M is a maximal ideal of R, and for every $r \in V \cap M$, C is projective with respect to the natural projection $h : E(S) \to E(S)/Ann_{E(S)}r$.*

Proof. (i) $\Longrightarrow$ (ii) Suppose that C is Ω_0-flat. Let $v : C \to E(S)/Ann_{E(S)}r$ be a homomorphism, where $S \cong R/M$ for some maximal ideal M of R and $r \in V \cap M$. Consider an exact sequence (1) with B projective. Then there exists a homomorphism $u : B \to E(S)$ such that $hu = vg$, i.e., the diagram (5) is commutative. Since the exact sequence (1) is Ω_0-pure, it follows by Theorem 3.2 that there exists a homomorphism $w : C \to E(S)$ such that $hw = v$, i.e., C is projective with respect to h.

(ii) $\Longrightarrow$ (i) Suppose that (ii) holds. Consider the exact sequence (1) and the commutative diagram (5). Since C is projective with respect to h, there exists a homomorphism $w : C \to E(S)$ such that $hw = v$. By Theorem 3.2, the exact sequence (1) is Ω_0-pure. Hence C is Ω_0-flat. □

Denote by $\mathcal{A}$ the class of Ω_0-flat R-modules.

COROLLARY 3.4. *The class $\mathcal{A}$ is closed under direct sums and direct summands.*

THEOREM 3.5. *Consider the exact sequence* (1). *Then:*

(i) *The class $\mathcal{A}$ is closed under extensions.*

(ii) *If $B \in \mathcal{A}$ and the exact sequence* (1) *is Ω_0-pure, then $C \in \mathcal{A}$.*

(iii) *If R is a hereditary ring and $B \in \mathcal{A}$, then $A \in \mathcal{A}$.*

Proof. Let $S \cong R/M$ for some maximal ideal M of R, let $r \in V \cap M$ and let $h : E(S) \to E(S)/Ann_{E(S)}r$ be the natural projection.

(i) Let $A, C \in \mathcal{A}$. By Theorem 3.3, A, C are projective with respect to h. Since $E(S)$ is injective, it follows by Lemma 3.1 that B is projective with respect to h. Now by Theorem 3.3, B is Ω_0-flat, i.e., $B \in \mathcal{A}$.

(ii) Let $B \in \mathcal{A}$ and assume that the exact sequence (1) is Ω_0-pure. Also, let $v : C \to E(S)/Ann_{E(S)}r$ be a homomorphism. Since B is Ω_0-flat, it follows by Theorem 3.3 that there exists a homomorphism $u : B \to E(S)$ such that $vg = hu$. Thus we obtain a commutative diagram of R-modules (5). But the exact sequence (1) is Ω_0-pure, hence by Theorem 3.2 there exists a homomorphism $w : C \to E(S)$ such that $hw = v$. Therefore, C is projective with respect to h. Now by Theorem 3.3, C is Ω_0-flat, i.e., $C \in \mathcal{A}$.

(iii) Let R be hereditary, let $B \in \mathcal{A}$ and let $\alpha : A \to E(S)/Ann_{E(S)}r$ be a homomorphism. Then $E(S)/Ann_{E(S)}r$ is injective, hence there exists a homomorphism $\beta : B \to E(S)/Ann_{E(S)}r$ such that $\beta f = \alpha$. We obtain a commutative diagram of R-modules with exact rows

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} C \longrightarrow 0 \\ & & & & \downarrow{\scriptstyle\alpha} & \swarrow{\scriptstyle\beta} & & \\ E(S) & \xrightarrow{h} & & & E(S)/Ann_{E(S)}r & \longrightarrow & 0 & \end{array}$$

By Theorem 3.3, there exists a homomorphism $\gamma : B \to E(S)$ such that $h\gamma = \beta$, because B is Ω_0-flat. Hence $h\gamma f = \beta f = \alpha$. Therefore, A is projective with respect to h. Now, by Theorem 3.3, $A \in \mathcal{A}$. □

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