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MULTI-VALUED MAPPINGS ON METRIC SPACES

LJUBOMIR B. ĆIRIĆ and JEONG S. UME

Abstract. We consider a multi-valued mapping F of a complete metric space (X, d) into the class $B(X)$ of nonempty, bounded subsets of X. For A, B in $B(X)$ we define $\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$.

It is proved that if F satisfies the contractive type condition $\delta(Fx, Fy) \leq \max\{\varphi_1(d(x, y)), \varphi_2(\delta(x, Fx)), \varphi_3(\delta(y, Fy)), \varphi_4(\delta(x, Fy)), \varphi_5(\delta(y, Fx))\}$ for all $x, y \in X$, where $\varphi_j : [0, +\infty) \to [0, +\infty)$, $j \in \{1, 2, 3, 4, 5\}$, are real functions satisfying: (a) $\varphi_j(t) < t$ for $t > 0$, (b) $\lim_{s \to t+} \varphi_j(s) < t$ for $t > 0$, (c) φ_j are nondecreasing and (d) $\lim_{t \to +\infty} (t - \varphi_j(t)) = +\infty$, then there exists a unique point z in X such that $Fz = \{z\}$. This result is a generalization of known results in this area and include, as special cases some theorems of Fisher, Khan and Kubiaczyk, Reich, Ćirić and Rhoades and Watson.

Key words. Complete metric spaces, fixed points, multi-valued mappings.

1. INTRODUCTION

In the fixed point theory for multi-valued mappings some theorems require that the range of each point to be compact, other bounded. In some cases the contractive conditions involve the Hausdorff metric induced by the metric d, in others the diameter of sets. Such is the case in this paper. The contractive condition considered here is a substantial generalization of the contractive conditions studied by Reich [9], Ćirić [1] and Fisher [5], as well as of the contractive definitions considered by Khan and Kubiaczyk [6] and by Rhoades and Watson [10].

Throughout the paper (X, d) denotes a complete metric space and $B(X)$ is the set of all nonempty, bounded subsets of X. For A, B in $B(X)$ the function $\delta(A, B)$ is defined by

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}.$$

For $\delta(\{a\}, B), \delta(A, \{b\})$ and $\delta(\{a\}, \{b\})$ we write $\delta(a, B), \delta(A, b)$ and $d(a, b)$, respectively. It follows easily from the definition that $\delta(B, A) = \delta(A, B) \geq 0$ and $\delta(A, C) \leq \delta(A, B) + \delta(B, C)$ for all A, B, C in $B(X)$. For any subsets A, B of X the distance between A and B is defined by

$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

For $D(\{a\}, B)$ we write $D(a, B)$.

A multi-valued mapping F on a set X has a fixed point $x \in X$ if $x \in Fx$. If $Fx = \{x\}$, then x is called a stationary point (or a strict fixed point) of F.

In [1] Ćirić defined and considered a mapping $F : X \to B(X)$ which satisfies the following contractive condition

(1) $\delta(Fx, Fy) \leq c \max\{d(x,y), \delta(x, Fx), \delta(y, Fy), D(x, Fy), D(y, Fx)\}$

for all x, y in X, where $0 \leq c < 1$.

Generalizing Theorem 2 in Ćirić [1], Fisher [5] proved the following theorem.

THEOREM 1.1. (Fisher [5, Theorem 2]). *Let F be a mapping of (X, d) into $B(X)$ satisfying the inequality*

(2) $\delta(Fx, Fy) \leq c \max\{d(x,y), \delta(x, Fx), \delta(y, Fy), \delta(x, Fy), \delta(y, Fx)\}$

for all x, y in X, where $0 \leq c < 1$. If F also maps $B(X)$ into itself, that is $F(A) = U_{a \in A} Fa \in B(X)$ for each $A \in B(X)$, then F has a unique fixed point z in X and further $F(z) = \{z\}$.

The added condition in Theorem 1.1, namely that $F(A)$ is bounded is strong and also may be difficult to test. So it is of interest to delete it. Using an addapted method we shall prove a fixed point theorem without such hypotheses, even if F satisfies substantial more general contractive condition than (2).

We need the following Lemma of Matkowski [7] and Singh and Meade [11].

LEMMA 1.1. *Let $\varphi : [0, +\infty) \to [0, +\infty)$ be a right continuous real function satisfying $\varphi(t) < t$ for $t > 0$. Then*

$$\lim_{n \to \infty} \varphi^n(t) = 0 \quad \text{for} \quad t > 0,$$

where φ^n is the n-th iteration of φ.

2. MAIN RESULT

Throughout the paper by Φ we denote the collection of functions $\varphi : [0, +\infty) \to [0, +\infty)$ which have the following properties:

(a) $\varphi(t) < t$ for all $t > 0$,

(b) $\lim_{s \to t+} \varphi(s) < t$ for all $t > 0$,

(c) $\varphi(t)$ is nondecreasing,

(d) $\lim_{t \to +\infty} (t - \varphi(t)) = +\infty$.

LEMMA 2.1. *If φ_1, $\varphi_2 \in \Phi$ then there is a $\varphi \in \Phi$ such that*

$$\varphi_1(t), \varphi_2(t) \leq \varphi(t) \quad \text{for all} \quad t > 0.$$

Proof. Define $\varphi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = \max\{\varphi_1(t), \varphi_2(t)\}$. Then from Lemma in [2] it follows that φ has properties (a), (b) and (c). To show that φ satisfies (d), let $E > 0$ be arbitrary. Since φ_1 and φ_2 satisfy (d), there exist $\Delta_j = \Delta_j(E) > 0$, $j \in \{1, 2\}$, such that

$$t - \varphi_1(t) > E \quad \text{for all} \quad t > \Delta_1, \qquad t - \varphi_2(t) > E \quad \text{for all} \quad t > \Delta_2.$$

Set $\Delta = \max\{\Delta_1, \Delta_2\}$. Then for all $t > \Delta$ we have

$$t - \varphi(t) = t - \max\{\varphi_1(t), \varphi_2(t)\} = \min\{(t - \varphi_1(t)), (t - \varphi_2(t))\} > E.$$

Thus φ also possess the property (d). The proof of Lemma is complete. □

Now we shall prove the following result:

THEOREM 2.1. *Let (X, d) be a complete metric space and let $F : X \to B(X)$ be a multi-valued mapping satisfying*

$$\begin{aligned}\delta(Fx, Fy) \quad \le \quad & \max\{\varphi_1(d(x,y)), \varphi_2(\delta(x,Fx)), \varphi_3(\delta(y,Fy)), \\ & \varphi_4(\delta(x,Fy)), \varphi_5(\delta(y,Fx))\}\end{aligned} \tag{3}$$

for all x, y in X, where $\varphi_j \in \Phi$, $j \in \{1,2,3,4,5\}$. Then F has a unique stationary point in X.

Proof. Let x_0 in X be arbitrary. Define a sequence $\{x_n\}$ in X as follows. Since now Fx_0 is defined, pick x_1 in Fx_0. Now Fx_1 is defined and let x_2 be any fixed point in Fx_1. Then we have that Fx_2 is well defined and let x_3 in Fx_2 be arbitrary. Continuing in this manner we inductively define two sequences: $\{x_n\}$ in X and $\{Fx_n\}$ in $B(X)$ such that

$$x_n \in Fx_{n-1} \quad (n = 1, 2, \ldots), \tag{4}$$

where x_n is arbitrary fixed point in Fx_{n-1}, nothing else.

We shall show that

$$\sup\{\delta(x_r, Fx_s) : x_r \in \{x_n\}, Fx_s \in \{Fx_n\}\} < +\infty, \tag{5}$$

where $\varphi \in \Phi$ is such that

$$\varphi_1(t), \varphi_2(t), \varphi_3(t), \varphi_4(t), \varphi_5(t) \le \varphi(t) \quad \text{for all} \quad t > 0. \tag{6}$$

Such φ exists from an extended version of Lemma 2.1.

First we prove that for any fixed positive integer n we have

$$\max\{\delta(x_r, Fx_s) : r, s = 0, 1, \ldots, n\} = \delta(x_0, Fx_k) \tag{7}$$

for some $k = k(n) \le n$. Suppose the contrary. Then there is $p \ge 1$ such that

$$\delta(x_p, Fx_k) = \max\{\delta(x_r, Fx_s) : 0 \le r, s \le n\}. \tag{8}$$

We may assume that $\delta(x_p, Fx_k) > 0$ for each n, since otherwise $Fx_0 = \{x_0\}$ and we have finished the proof.

From (3), as $x_p \in Fx_{p-1}$, we have

$$\begin{aligned}\delta(x_p, Fx_k) &\le \delta(Fx_{p-1}, Fx_k) \\ &\le \max\{\varphi_1(d(x_{p-1}, x_k)), \varphi_2(\delta(x_{p-1}, Fx_{p-1})), \varphi_3(\delta(x_k, Fx_k)), \\ &\quad \varphi_4(\delta(x_{p-1}, Fx_k)), \varphi_5(\delta(x_k, Fx_{p-1}))\}.\end{aligned} \tag{9}$$

From this and (6) we have

$$\begin{aligned}\delta(x_p, Fx_k) \le \max\{&\varphi(d(x_{p-1}, x_k)), \varphi(\delta(x_{p-1}, Fx_{p-1})), \\ &\varphi(\delta(x_k, Fx_k)), \varphi(\delta(x_{p-1}, Fx_k)), \varphi(\delta(x_k, Fx_{p-1}))\}.\end{aligned} \tag{10}$$

Since φ is nondecreasing, from (10) and (8) we get $\delta(x_p, Fx_k) \le \varphi(\delta(x_p, Fx_k))$. Hence, by (a) we have $\delta(x_p, Fx_k) < \delta(x_p, Fx_k)$, a contradiction. Therefore, x_p must be x_0. Thus we proved (7).

For any positive integer n set

$$t_n = \delta(x_0, Fx_k), \tag{11}$$

where $k = k(n)$ is chosen such that (7) holds. Since by the triangle inequality

$$t_n = \delta(x_0, Fx_k) \le \delta(x_0, Fx_0) + \delta(Fx_0, Fx_k),$$

from (3), (6) and (11) we obtain $t_n \le \delta(x_0, Fx_0) + \max\{\varphi(d(x_0, x_k)), \varphi(\delta(x_0, Fx_0)), \varphi(\delta(x_k, Fx_k)), \varphi(\delta(x_0, Fx_k)), \varphi(\delta(x_k, Fx_0))\} \le \delta(x_0, Fx_0) + \varphi(t_n)$. Hence we get

$$t_n - \varphi(t_n) \le \delta(x_0, Fx_0). \tag{12}$$

From definition of t_n (see (7)), it follows that $\{t_n\}$ is nondecreasing sequence. Therefore, $\lim\limits_{n\to\infty} t_n$ exists. If we suppose that $\lim\limits_{n\to\infty} t_n = +\infty$, then the right-hand side of (12) is bounded, but from hypothesis (d) for φ, the left-hand side is unbounded, which is a contradiction. Therefore, we proved (5).

Now we shall show that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$ be arbitrary. Set

$$L = \sup\{\delta(x_r, Fx_s) : r, s \ge 0\}.$$

From (5), L is finite number and by Lemma 1.1 there is a positive integer N such that

$$\varphi^N(L) < \varepsilon. \tag{13}$$

From (3) and (7) it follows that for $n \ge m \ge N$ we have, as $x_m \in Fx_{m-1}$,

$$\delta(x_m, Fx_n) \le \delta(Fx_{m-1}, Fx_n)) \le \varphi(\delta(x_{m-1}, Fx_k)), \tag{14}$$

where $m - 1 \le k \le n$. Since by the same arguments

$$\delta(x_{m-1}, Fx_k) \le \delta(Fx_{m-2}, Fx_k) \le \varphi(\delta(x_{m-2}, Fx_p)),$$

where $m - 2 \le p \le k$, by (14) we get

$$\delta(x_m, Fx_n) \le \varphi^2(\delta(x_{m-2}, Fx_p)); \quad m - 2 \le p \le k \le n.$$

Proceeding in this manner, we obtain

$$\delta(x_m, Fx_n) \le \varphi^m(\delta(x_0, Fx_q)); \quad 0 \le q \le n. \tag{15}$$

Since φ is nondecreasing and $x_{n+1} \in Fx_n$, by (15) and (13) we have

$$d(x_m, x_{n+1}) \le \delta(x_m, Fx_n) \le \varphi^m(\delta(x_0, Fx_q)) \le \varphi^m(L) \le \varphi^N(L) < \varepsilon. \tag{16}$$

From (16) we conclude that $\{x_n\}$ is a Cauchy sequence. Also from (16) we conclude that a sequence of reals $\{\delta(x_n, Fx_n)\}$ tends to zero when n tends to infinity.

Since X is complete, $\{x_n\}$ converges to some point, say z in X. Suppose, by way of contradiction, that $\delta(z, Fz) > 0$. Using the triangle inequality and (6), from (3) we have

$$\delta(z, Fz) \leq d(z, x_{n+1}) + \delta(Fx_n, Fz) \leq d(z, x_{n+1})$$
$$+ \max\{\varphi(d(x_n, z)), \varphi(\delta(x_n, Fx_n)), \varphi(\delta(z, Fz)), \varphi(\delta(x_n, Fz)), \varphi(\delta(z, Fx_n))\}.$$

Hence, as φ is nondecreasing, by the triangle inequality we get

$$\delta(z, Fz) \leq d(z, x_{n+1}) + \varphi(d(x_n, z) + \delta(x_n, Fx_n) + \delta(z, Fz)). \tag{17}$$

Since $\delta(x_n, Fx_n) \to 0$ and $d(z, x_n) \to 0$ as $n \to \infty$ we have that

$$[\delta(z, Fz) + d(z, x_n) + \delta(x_n, Fx_n)] \to \delta(z, Fz) \tag{18}$$

when n tends to infinity. Taking the limit of both sides in (14) when n tends to infinity, by (18) and from (b) we have

$$\delta(z, Fz) \leq \lim_{n\to\infty} \varphi[\delta(z, Fz) + d(z, x_n) + \delta(x_n, Fx_n)] < \delta(z, Fz),$$

a contradiction. Therefore, $\delta(z, Fz) = 0$. Hence $Fz = \{z\}$. The uniqueness of a strict fixed (stationary) point is implied by (3). The proof of the theorem is complete. □

Remark 2.1. Theorem 2.1. with $\varphi_j(t) = c \cdot t$, $0 < c < 1$, $j = 1, 2, 3, 4, 5$, is a generalization of the corresponding theorems of Reich [9], Ćirić [1] and Fisher [5]. Theorem 2.1. is also a generalization of Theorem 1 in Khan and Kubiaczyk [6] and Theorem 2 in Rhoades and Watson [9].

Remark 2.2. The following example shows that the contractive condition (3) is substantial more general then the condition (2), even if (X, d) is compact and convex Euclidean space.

Example. Let $X = [0, \frac{1}{2}]$ be the closed interval with usual metric and let $F : X \to B(X)$ and $\varphi : [0, +\infty) \to [0, +\infty)$ be mappings defined as follows:

$$\begin{aligned} Fx &= [x - x^2, x - x^3] \quad \text{for all} \quad 0 \leq x \leq \frac{1}{2}, \\ \varphi(t) &= t - t^3, \ \text{ if } 0 \leq t \leq \frac{1}{2}, \qquad \varphi(t) = \frac{3}{4}t, \ \text{ if } t > \frac{1}{2}, \end{aligned}$$

respectively. Let x, y in X be arbitrary. Without loss of generality we may suppose that $x \leq y$. Then we have

$$\begin{aligned} \delta(Fx, Fy) &= y - y^3 - x + x^2, \\ M(x, y) &= \max\{d(x, y), \delta(x, Fx), \delta(y, Fy), \delta(x, Fy), \delta(y, Fx)\} = \delta(y, Fx), \\ \delta(y, Fx) &= y - x(1 - x). \end{aligned}$$

Since φ is increasing, from (3), with $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \varphi_5 = \varphi$ we have, as $x(1 - x) \geq 0$ implies that $-y \leq -(y - x(1 - x))$,

$$\begin{aligned} \delta(Fx, Fy) &= y - y^3 - x + x^2 = (y - x(1 - x)) - y^3 \\ &\leq (y - x(1 - x)) - (y - x(1 - x))^3 = \varphi(\delta(y, Fx)). \end{aligned}$$

Thus, F satisfies (3) and we can apply our Theorem 2.1. On the other hand, for any fixed c; $0 < c < 1$, we have, for $x = 0$ and each $y \in X$ with $0 < y < \sqrt{1-c}$,

$$\delta(F0, Fy) = (1 - y^2)y > c \cdot y = c\delta(y, F0) = c \cdot M(0, y).$$

Therefore, F does not satisfy (2).

Note that further generalization of Theorem 2.1 in light of result in [3–4], [8] and [6, Theorem 3] may be of interest.

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Faculty of Mechanical Engineering
27.March 80, Beograd, Serbia
E-mail: ciric@alfa.mas.bg.ac.yu

Changwon National University
Dept. of Applied Mathematics
Changwon 641-773, Korea
E-mail: jsume@sarim.changwon.ac.kr