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MORSE INDEX OF HARMONIC MAPS INTO HP^n

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Abstract. In this paper, the index and the nullities of harmonic maps into HP^n are calculated.

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1. INTRODUCTION

It is well known that every non-constant harmonic map ϕ to or from S^m $(m \geq 3)$ is unstable, that is, $\mathrm{Ind}_E(\phi) \geq 1$ ([9], [7]). It's natural to study its unstability. The first step in this direction was given by A. El Soufi, where he obtained its lower bound ([2], [3], [4]). In addition, he also investigated the case when the target manifold is CP^n. Recently, this method has been successful used in the study of the Morse index of Yang-Mills connection over unit spheres by S. Nayatani and H. Urakawa [8]. In this paper, we will deal with the Morse index of harmonic map $\phi\colon S^m \to HP^n$.

Let (M, g) be an m-dimensional compact Riemannian manifold without boundary and (N, h) an n-dimensional Riemannian manifold. We denote by D and $\bigtriangledown$ the Levi-Civita connection on (M, g) and (N, h), respectively. A smooth map $\phi\colon (M, g) \to (N, h)$ is said to be *harmonic* if it is a critical point of the energy $E(\phi)$ defined by

$$E(\phi) = \int_M e(\phi) v_g$$
$$e(\phi) = \frac{1}{2}\sum_{i=1}^{m} h(\mathrm{d}\phi(e_i), \mathrm{d}\phi(e_i)),$$

where $\mathrm{d}\phi$ is the differential of ϕ. Namely, for every vector field V along ϕ

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} E(\phi_t) = 0.$$

Here $\phi_t\colon M \to N$ is an one parameter family of smooth maps with $\phi_0 = \phi$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \phi_t(x) = V_x \in T_{\phi(x)}N$$

for every point $x \in M$.

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The first and second variation formulas of the energy $E(\phi)$ for a harmonic map ϕ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} E(\phi_t) = -\int_M \langle \tau(\phi), V\rangle v_g \,,$$

$$H_\phi(V) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} E(\phi_t) = -\int_M \langle V, J_\phi V\rangle v_g \,.$$

Here $\tau(\phi) = \mathrm{tr}_g \bigtriangledown^\phi \mathrm{d}\phi$ is tensor field of ϕ and J_ϕ is a differential operator (called the *Jacobi operator*) acting on the space $\Gamma(\phi)$ of sections of the induced bundle $\phi^{-1}TN$. The operator J_ϕ is of the form

$$J_\phi V = -\mathrm{tr}_g \bigtriangledown^\phi \bigtriangledown^\phi V - \mathrm{Ric}^\phi(V)\,,$$

$$\mathrm{Ric}^\phi(V) = \mathrm{tr}_g R^N(\mathrm{d}\phi, V)\mathrm{d}\phi.$$

The Morse index and nullity of ϕ are defined by

$$\mathrm{Ind}_E(\phi) = \sup\{\dim F : F \subset \Gamma(\phi) \ \text{ and } \ H_\phi \ \text{ is negative definite on } \ F\}$$

$$\mathrm{Nul}_E(\phi) = \dim(\ker H_\phi).$$

Let $N_-(\phi)$ $(N_0(\phi))$ be the number of negative (zero) eigenvalue of Jacobi operator J_ϕ. Then we have

$$\mathrm{Ind}_E(\phi) = N_-(J_\phi), \quad \mathrm{Nul}_E(\phi) = N_0(J_\phi)$$

For volume function $V(\phi)$, we can also define $\mathrm{Ind}_V(\phi)$ and $\mathrm{Nul}_V(\phi)$.

LEMMA 1. *Let $\phi\colon (M,g) \to HP^n$ be a harmonic map. Then for $\forall V \in \Gamma(\phi)$, we have*

$$J_\phi(J_k V) = J_k J_\phi V + \mathrm{tr}_g(\langle J_k V, \mathrm{d}\phi\rangle \mathrm{d}\phi - \langle V, \mathrm{d}\phi\rangle J_k \mathrm{d}\phi).$$

Here $\{J_1, J_2, J_3\}$ is a quaternionic Kahler structure of HP^n. In particular, if ϕ is weakly conformal, then we have

$$J_\phi(J_k \mathrm{d}\phi(X)) = J_k J_\phi(\mathrm{d}\phi(X)) - \frac{|\mathrm{d}\phi|^2}{m}(J_k \mathrm{d}\phi(X))^\perp$$

for any vector field $X \in \Gamma(M)$.

Proof. Since J_k commutes with $\bigtriangledown^\phi$, we have, for $\forall V \in \Gamma(\phi)$,

$$\begin{aligned} J_\phi(J_k V) - J_k J_\phi V &= -\mathrm{tr}_g \bigtriangledown^\phi \bigtriangledown^\phi J_k V - \mathrm{Ric}^\phi(J_k V) \\ &\quad + J_k(\mathrm{tr}_g \bigtriangledown^\phi \bigtriangledown^\phi V + \mathrm{Ric}^\phi(V)) \\ &= J_k \mathrm{Ric}^\phi(V) - \mathrm{Ric}^\phi(J_k V)\,. \end{aligned}$$

On the other hand, the curvature tensor of HP^n is given by

$$R(X,Y)X = \frac{1}{4}\left(|X|^2 Y - \langle X, Y\rangle X - 3\sum_{i=1}^{3}\langle X, J_i Y\rangle J_i X\right).$$

So we get

$$\begin{aligned}\mathrm{Ric}^{\phi}(V) &= \mathrm{tr}_g R(\mathrm{d}\phi, V)\mathrm{d}\phi \\ &= \frac{1}{4}\left[\left|\mathrm{d}\phi\right|^2 V - \mathrm{tr}_g(\langle V, \mathrm{d}\phi\rangle \mathrm{d}\phi + 3\sum_{i=1}^{3}\langle J_i V, \mathrm{d}\phi\rangle J_i \mathrm{d}\phi)\right]\end{aligned}$$

and

$$\begin{aligned}&J_k \mathrm{Ric}^{\phi}(V) - \mathrm{Ric}^{\phi}(J_k V) \\ &\quad= \frac{1}{4}\mathrm{tr}_g\Big(\langle J_k V, \mathrm{d}\phi\rangle \mathrm{d}\phi + 3\sum_{i=1}^{3}\langle J_i J_k V, \mathrm{d}\phi\rangle J_i \mathrm{d}\phi \\ &\qquad - \langle V, \mathrm{d}\phi\rangle J_k \mathrm{d}\phi - 3\sum_{i=1}^{3}\langle J_i V, \mathrm{d}\phi\rangle J_k J_i \mathrm{d}\phi\Big) \\ &\quad= \mathrm{tr}_g(\langle J_k V, \mathrm{d}\phi\rangle \mathrm{d}\phi - \langle V, \mathrm{d}\phi\rangle J_k \mathrm{d}\phi), \quad k = 1, 2, 3.\end{aligned}$$

If ϕ is weakly conformal, we consider the set

$$\Omega = \{x \in M : \mathrm{d}\phi(x) \neq 0\}.$$

Let $\{e_1, \cdots, e_m\}$ be a local orthogonal frame field at $x \in \Omega$. Then the set $\{\sqrt{m}\left|\mathrm{d}\phi\right|^{-1}\mathrm{d}\phi(e_i)\}_{i=1}^{m}$ is an orthogonal frame field and

$$\mathrm{tr}_g\langle J_k V, \mathrm{d}\phi\rangle \mathrm{d}\phi = \sum_{i=1}^{3}\langle J_k V, \mathrm{d}\phi(e_i)\rangle \mathrm{d}\phi(e_i) = \frac{\left|\mathrm{d}\phi\right|^2}{m}(J_k V)^{\perp},$$

$$\mathrm{tr}_g\langle V, \mathrm{d}\phi\rangle J_k \mathrm{d}\phi = J_k \sum_{i=1}^{3}\langle V, \mathrm{d}\phi(e_i)\rangle \mathrm{d}\phi(e_i) = \frac{\left|\mathrm{d}\phi\right|^2}{m}(J_k V)^{\perp}.$$

Setting $V = \mathrm{d}\phi(X)$ in above, we get

$$J_\phi(J_k \mathrm{d}\phi(X)) = J_k J_\phi(\mathrm{d}\phi(X)) - \frac{\left|\mathrm{d}\phi\right|^2}{m}(J_k \mathrm{d}\phi(X))^{\perp},$$

which finishes the proof of the lemma. □

LEMMA 2. [5] *For all $X \in \Gamma(M)$, we have*

$$J_\phi(\mathrm{d}\phi(X)) = \mathrm{d}\phi(J_I X) - 2\mathrm{tr}_g \nabla^{\phi} \mathrm{d}\phi(D.X, \cdot)\ ,$$

where I is identical map on M.

LEMMA 3. [3] *If $\phi\colon M \to N$ is totally geodesic and homothetic, then we have*

$$\mathrm{Ind}_E(\phi) - \mathrm{Ind}_V(\phi) = \mathrm{Ind}_E(I),$$
$$\mathrm{Nul}_E(\phi) - \mathrm{Nul}_V(\phi) = \mathrm{Nul}_E(I).$$

Now we consider the standard immersion $j_m\colon S^m \hookrightarrow HP^m$ (i.e., the composition of $RP^m \hookrightarrow CP^m \hookrightarrow HP^m$ and double cover $S^m \to RP^m$). In fact, j_m is a totally geodesic and totally real homothetic immersion.

PROPOSITION 4. *For the standard immersion $j_m\colon S^m \hookrightarrow HP^m$, we have*

$$\mathrm{Ind}_E(j_m) = \begin{cases} (m+1)(3m+8)/2 \ , & m \geq 3 \\ 18 \ , & m = 2 \end{cases}$$

$$\mathrm{Nul}_E(j_m) = \begin{cases} m(2m+5) \ , & m \geq 3 \\ 36 \ , & m = 2 \end{cases}$$

$$\mathrm{Ind}_V(j_m) = 3(m+1)(m+2)/2$$

$$\mathrm{Nul}_V(j_m) = \begin{cases} 3m(m+3)/2 \ , & m \geq 3 \\ 30 \ , & m = 2 \end{cases}$$

Proof. Since j_m is homothetic, totally geodesic and totally real, we have

$$\Gamma^{\perp}(j_m) = \oplus_{k=1}^{3} J_k \Gamma^{\top}(j_m)$$

and

$$\mathrm{Ind}_V(j_m) = N_-(J_{j_m}\Big|\Gamma^{\perp}(j_m)) = \sum_{k=1}^{3} N_-(J_{j_m}\Big|J_k\Gamma^{\top}(j_m))$$

For $V = J_k(\mathrm{d}j_m(X)) \in J_k\Gamma^{\top}(j_m)$, we have, from Lemma 1 and Lemma 2,

$$\begin{aligned} J_{j_m}(J_k \mathrm{d}j_m(X)) &= J_k(J_{j_m}\mathrm{d}j_m(X)) - 4J_k\mathrm{d}j_m(X) \\ &= J_k(\mathrm{d}j_m(J_I X)) - 4J_k\mathrm{d}j_m(X) \\ &= J_k\mathrm{d}j_m(J_I X - 4X). \end{aligned}$$

On the other hand, by [3, (28)], we have

$$J_I X = \triangle X - 2(m-1)X,$$

where $\triangle$ is Hodge-Laplace operator on S^m. So we obtain

$$J_{j_m}(J_k\mathrm{d}j_m(X)) = J_k\mathrm{d}j_m(\triangle X - 2(m-1)X), \quad k = 1,2,3.$$

Then

$$\mathrm{Ind}_V(j_m) = 3N_-(\triangle - 2(m+1)),$$

$$\mathrm{Nul}_V(j_m) = 3N_0(\triangle - 2(m+1)).$$

From [3] or [6], we know that, for $m \geq 3$,

$$\begin{aligned} &\lambda_1(\triangle) = m \ , && m(\lambda_1) = m+1 \\ &\lambda_2(\triangle) = 2(m-1) \ , && m(\lambda_2) = m(m+1)/2 \\ &\lambda_3(\triangle) = 2(m+1) \ , && m(\lambda_3) = m(m+3)/2 \end{aligned}$$

and for $m = 2$,

$$\begin{aligned} &\lambda_1(\triangle) = 2 \ , && m(\lambda_1) = 6 \\ &\lambda_2(\triangle) = 6 \ , && m(\lambda_2) = 10, \end{aligned}$$

where $m(\lambda)$ denotes the multiplicity of λ. From this, we get

$$\mathrm{Ind}_V(j_m) = 3(m+1+m(m+1)/2) = 3(m+1)(m+2)/2$$

$$\mathrm{Nul}_V(j_m) = \begin{cases} 3m(m+3)/2 \ , & m \geq 3 \\ 30 \ , & m = 2 \end{cases}$$

Using Lemma 3 and the following equalities (see [3]),

$$\mathrm{Ind}_E(I) = \begin{cases} m+1 \ , & m \geq 3 \\ 0 \ , & m = 2 \end{cases}$$

$$\mathrm{Nul}_E(I) = \begin{cases} m(m+1)/2 \ , & m \geq 3 \\ 6 \ , & m = 2 \ , \end{cases}$$

we obtain

$$\mathrm{Ind}_E(j_m) = \begin{cases} (m+1)(3m+8)/2 \ , & m \geq 3 \\ 18 \ , & m = 2 \end{cases}$$

$$\mathrm{Nul}_E(j_m) = \begin{cases} m(2m+5) \ , & m \geq 3 \\ 36 \ , & m = 2 \ . \end{cases}$$

2. MAIN THEOREMS AND THEIR PROOFS

THEOREM 5. *For any harmonic totally real immersion $\phi\colon S^m \to HP^n$, $(m \geq 3)$, we have*

$$\mathrm{Ind}_E(\phi) \geq \mathrm{Ind}_E(j_m) = (m+1)(3m+8)/2.$$

Proof. Consider the sets

$$F_1 = \{\mathrm{d}\phi(\overline{a}): \ a \in R^{m+1}\}, \ \ F_{2k} = J_k \mathrm{d}\phi(A), \ \ F_{3k} = J_k \mathrm{d}\phi(K),$$

where $A = \{\overline{a} : a \in R^{m+1}\}$, $\overline{a}(x) = a - \langle a, x\rangle x$, and K is the space of Killing vector field. By [5, 5.4] we have

$$J_\phi(\mathrm{d}\phi(\overline{a})) = \frac{2-m}{|a|}\mathrm{d}\phi(\overline{a}), \ \ \overline{a} \in A$$

$$H_\phi(\mathrm{d}\phi(\overline{a})) = \frac{2-m}{|a|}\int_{S^m} \left|\mathrm{d}\phi(\overline{a})\right|^2 \mathrm{d}v < 0$$

It's easy to know that, if $\mathrm{d}\phi(\overline{a}) = 0$ with $a \neq 0$, then ϕ is constant. So H_ϕ is negative definite on F_1. Similarly, from Lemma 1, we have

$$\begin{aligned} J_\phi(J_k \mathrm{d}\phi(\overline{b})) &= J_k J_\phi(\mathrm{d}\phi(\overline{a})) \\ &\quad + \sum_{i=1}^{3} (\langle J_k(\mathrm{d}\phi(\overline{b})), \mathrm{d}\phi(e_i)\rangle \mathrm{d}\phi(e_i) - \langle \mathrm{d}\phi(\overline{a}), \mathrm{d}\phi(e_i)\rangle J_k \mathrm{d}\phi(e_i)) \\ &= \frac{2-m}{|b|} J_k \mathrm{d}\phi(\overline{a}) - \sum_{i=1}^{3} \langle \mathrm{d}\phi(\overline{b}), \mathrm{d}\phi(e_i)\rangle J_k \mathrm{d}\phi(e_i) \\ H_\phi(J_k \mathrm{d}\phi(\overline{b})) &= \frac{2-m}{|b|} \int_{S^m} \left|\mathrm{d}\phi(\overline{b})\right|^2 \mathrm{d}v - \sum_{i=1}^{3} \int_{S^m} \langle \mathrm{d}\phi(\overline{b}), \mathrm{d}\phi(e_i)\rangle^2 \mathrm{d}v \le 0. \end{aligned}$$

By the same discussion as above, we know that H_ϕ is negative definite on F_{2k} ($k = 1, 2, 3$). For $X \in K$, by the same calculation, we have

$$\begin{aligned} J_\phi(J_k \mathrm{d}\phi(X)) &= \cdots = -\sum_{i=1}^{3} \langle \mathrm{d}\phi(X), \mathrm{d}\phi(e_i)\rangle J_k \mathrm{d}\phi(e_i), \\ H_\phi(J_k \mathrm{d}\phi(X)) &= -\sum_{i=1}^{3} \int_{S^m} \langle \mathrm{d}\phi(X), \mathrm{d}\phi(e_i)\rangle^2 \mathrm{d}v \le 0 \end{aligned}$$

and

$$H_\phi(J_k \mathrm{d}\phi(X)) = 0 \iff \mathrm{d}\phi(X) = 0 \iff X = 0.$$

So H_ϕ is negative definite on F_{3k} ($k = 1, 2, 3$). Since $\mathrm{d}\phi(A)$ and $\mathrm{d}\phi(K)$ are the eigenspace of J_ϕ with eigenvalue $2 - m$ and 0, respectively, and, ϕ is totally real, we know that H_ϕ is negative on

$$F = \mathrm{d}\phi(A) \oplus \sum_{i=1}^{3} (J_k \mathrm{d}\phi(A) \oplus J_k \mathrm{d}\phi(K)),$$

that is,

$$\begin{aligned} \mathrm{Ind}_E(\phi) &\ge \dim F = 4(m+1) + 3m(m+1)/2 \\ &= (m+1)(3m+8)/2 = \mathrm{Ind}_E(j_m). \end{aligned}$$

Here we use $\dim F_1 = \dim \mathrm{d}\phi(A) = m + 1$ (see [3]). □

For the index of volume variation, we have analogous result.

THEOREM 6. *For any homothetic totally real minimal immersion*

$$\phi\colon (M, g) \to HP^n,$$

we have

$$\mathrm{Ind}_V(\phi) \ge \mathrm{Ind}_V(j_m) = (m+1)(m+2)/2.$$

From the following Lemma, we can obtain the proof of this theorem immediately.

LEMMA 7. *Let $\phi\colon (M,g) \to HP^n$ is a homothetic totally real minimal immersion. Then we have*

$$\min(\mathrm{Ind}_E(\phi), \mathrm{Ind}_V(\phi)) \geq 3(\mathrm{Ind}_E(I) + \mathrm{Nul}_E(I)),$$

where I is identity map on M.

Proof. Let F be a set of $\Gamma(I)$ on which H_I is non-positive, that is, $H_I(X) \leq 0$ for all $X \in F$). So we have $\dim F = \mathrm{Ind}_E(I) + \mathrm{Nul}_E(I)$ and for $\forall X \in F$, by Lemma 1 and Lemma 2,

$$\begin{aligned}
\langle J_\phi(J_k \mathrm{d}\phi(X)), J_k \mathrm{d}\phi(X)\rangle &= \langle J_k(J_\phi \mathrm{d}\phi(X)) - \frac{|\mathrm{d}\phi|^2}{m}(J_k \mathrm{d}\phi(X))^\perp, J_k \mathrm{d}\phi(X)\rangle \\
&= \langle J_\phi \mathrm{d}\phi(X), \mathrm{d}\phi(X)\rangle - \frac{|\mathrm{d}\phi|^2}{m}\left|(J_k \mathrm{d}\phi(X))^\perp\right|^2 \\
&= \langle \mathrm{d}\phi(J_I X), \mathrm{d}\phi(X)\rangle - \frac{|\mathrm{d}\phi|^2}{m}\left|(J_k \mathrm{d}\phi(X))^\perp\right|^2 \\
&= \frac{|\mathrm{d}\phi|^2}{m}(\langle J_I X, X\rangle - \left|J_k \mathrm{d}\phi(X)\right|^2); \\
H_\phi(J_k \mathrm{d}\phi(X)) &= \frac{|\mathrm{d}\phi|^2}{m}\left(H_I(X) - \int_M \left|J_k \mathrm{d}\phi(X)\right|^2 v_g\right) \\
&\leq -\frac{|\mathrm{d}\phi|^2}{m}\int_M \left|J_k \mathrm{d}\phi(X)\right|^2 v_g, \quad k = 1, 2, 3\,.
\end{aligned}$$

By the definition, we get

$$\mathrm{Ind}_E(\phi) \geq \sum_{k=1}^{3} \dim J_k \mathrm{d}\phi(F) = 3\dim F = 3(\mathrm{Ind}_E(I) + \mathrm{Nul}_E(I)).$$

On the other hand, the second variation of volume function $V(\phi)$ along $V \in \Gamma^\perp(\phi)$ is

$$Q_\phi(V) = H_\phi(V) - 2\int_M \left|(\nabla^\phi V\right|^2 v_g \leq H_\phi(V)$$

It follows that $\mathrm{Ind}_V(\phi) \geq 3(\mathrm{Ind}_E(I) + \mathrm{Nul}_E(I))$. □

Finally, we have

THEOREM 8. *Let ϕ be a stable harmonic map from M to S^2. Then*

$$\mathrm{Nul}_E(\phi) \geq 10.$$

Proof. Let J be complex structure on S^2 and g the Lie algebra of Killing vector fields of S^2. Then for all $X \in g$, we have $J_\phi X = 0$ and $J_\phi JX = JJ_\phi X = 0$ (see [1]), that is, $H_\phi(JX, JX) = 0, \forall X \in g$. Since Killing vector field of S^2 correspond to the eigenspace w.r.t. eigenvalue $\lambda = 6$ which dimension is 10. We get $\mathrm{Nul}_E(\phi) \geq 10$. □

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