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LOCAL EXISTENCE OF SOLUTIONS TO A CLASS OF NONCONVEX SECOND ORDER DIFFERENTIAL INCLUSIONS

AURELIAN CERNEA

Abstract. We prove the local existence of solutions to the Cauchy problem $x'' \in F(x,x') + f(t,x,x')$, $x(0) = x_0, x'(0) = y_0$, where F is a set-valued map contained in the Fréchet subdifferential of a ϕ-convex function of order two and f is a Carathéodory single valued map.

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1. INTRODUCTION

In this paper we consider the Cauchy problem for second order differential inclusion

$$x'' \in F(x,x') + f(t,x,x'), \quad x(0) = x_0, \quad x'(0) = y_0, \tag{1.1}$$

where $F(.,.)$ is a given set-valued map, $f(.,.,.)$ is a given Carathéodory map and $x_0, y_0 \in R^n$.

Second order differential inclusions were studied by many authors, mainly in the case when the multifunction is convex valued. Several existence results may be found in [8], [10], [12], etc.

Recently, in [6], [7], [11], the situation when the multifunction is not convex valued is considered. More exactly, in [11] it is proved the existence of solutions of the problem

$$x'' \in F(x,x'), \quad x(0) = x_0,\ x'(0) = y_0, \tag{1.2}$$

when $F(.,.)$ is an upper semicontinuous compact valued multifunction contained in the subdifferential of a proper convex function. In [7] it is proved the existence of solutions of the problem (1.1) with F as in [11] and $f(.,.,.)$ is a Carathéodory map. In [6] the existence of solutions for problem (1.2) is obtained with $F(.,.)$ an upper semicontinuous compact valued multifunction contained in the Fréchet subdifferential of a ϕ-convex function of order two.

The aim of this paper is to unify the results quoted above by proving the existence of local solutions of the problem (1.1) when $F(.,.)$ is an upper semicontinuous compact valued multifunction contained in the Fréchet subdifferential of a ϕ-convex function of order two and $f(.,.,.)$ is a Carathéodory map. Since the class of proper convex functions is strictly contained in the class of ϕ-convex functions, our result generalizes the one in [7]. Our existence result contains Peano's existence theorem (for second order differential equations) as a particular case. On the other hand, our result may be considered as an extension of the previous result of Ancona and Colombo ([1]) obtained for first order differential inclusions of the form

$$x' \in F(x) + f(t,x), \quad x(0) = x_0, \tag{1.3}$$

with F a cyclically monotone set-valued map and f a Carathéodory map. The proof of our main result follows the general ideas in [1], [6] and [11].

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main result.

2. PRELIMINARIES

We denote by $\mathcal{P}(R^n)$ the set of all subsets of R^n and by R_+ the set of all positive real numbers. For $\epsilon > 0$ we put $B_\epsilon(x) = \{y \in R^n; ||y-x|| < \epsilon\}$. With B we denote the unit ball in R^n. By $cl(A)$ we denote the closure of the set $A \subset R^n$, by $co(A)$ we denote the convex hull of A and we put $||A|| = \sup\{||a||; a \in A\}$.

Let $\Omega \subset R^n$ be an open set and let $V : \Omega \to R \cup \{+\infty\}$ be a function with domain $D(V) = \{x \in R^n; V(x) < +\infty\}$.

Definition 2.1. The multifunction $\partial_F V : \Omega \to \mathcal{P}(R^n)$, defined as:

$$\partial_F V(x) = \{\alpha \in R^n, \liminf_{y\to x} \frac{V(y)-V(x)- < \alpha, y-x >}{||y-x||} \geq 0\} \text{ if } V(x) < +\infty$$

and $\partial_F V(x) = \emptyset$ if $V(x) = +\infty$ is called the Fréchet subdifferential of V.

We also put $D(\partial_F V) = \{x \in R^n; \partial_F V(x) \neq \emptyset\}$.

According to [9] the values of $\partial_F V$ are closed and convex.

Definition 2.2. Let $V : \Omega \to R \cup \{+\infty\}$ be a lower semicontinuous function. We say that V is a ϕ-convex of order 2 if there exists a continuous map $\phi_V : (D(V))^2 \times R^2 \to R_+$ such that for every $x, y \in D(\partial_F V)$ and every $\alpha \in \partial_F V(x)$ we have

$$V(y) \geq V(x) + < \alpha, x-y > - \phi_V(x,y,V(x),V(y))(1+||\alpha||^2)||x-y||^2.$$

In [9] there are several examples and properties of such maps.

In what follows, for $F : D \subset R^n \times R^n \to \mathcal{P}(R^n)$, $f : R \times D \to R^n$ and for any $(x_0, y_0) \in D$ we consider problem (1.1) under the following assumptions:

Hypothesis 2.3. i) *$D \subset R^n \times R^n$ is an open set and $F : D \to \mathcal{P}(R^n)$ is upper semicontinuous (i.e., $\forall z \in D$, $\forall \epsilon > 0$ there exists $\delta > 0$ such that $||z - z'|| < \delta$ implies $F(z') \subset F(z) + \epsilon B$) with compact values.*

ii) *There exists a proper lower semicontinuous ϕ-convex function of order two $V : R^n \to R \cup \{+\infty\}$ such that*

$$F(x,y) \subset \partial_F V(y), \quad \forall (x,y) \in D.$$

iii) *$f : R \times D \to R^n$ is Carathéodory, i.e. for every $(x, y) \in D$, $t \to f(t, x, y)$ is measurable, for a.e. $t \in R$ $(x, y) \to f(t, x, y)$ is continuous and there exists $p(.) \in L^2(R, R_+)$ such that*

$$||f(t, x, y)|| \leq p(t) \quad a.e. t \in R, \quad \forall (x, y) \in D.$$

Finally, by a solution of problem (1.1) we mean an absolutely continuous function $x(.) : [0, T] \to R^n$ with absolutely continuous derivative $x'(.)$ such that $x(0) = x_0$, $x'(0) = y_0$ and

$$x''(t) \in F(x(t), x'(t)) + f(t, x(t), x'(t)) \quad a.e. \ [0, T].$$

3. THE MAIN RESULT

Our main result is the following.

THEOREM 3.1. *Consider $F : D \to \mathcal{P}(R^n)$ and $f : R \times D \to R^n$ that satisfies Hypothesis 2.3. Then, for every $(x_0, y_0) \in D$ there exist $T > 0$ and $x(.) : [0, T] \to R^n$ solution to problem* (1.1).

Proof. Consider $(x_0, y_0) \in D$. Since D is open, there exists $R > 0$ such that $\overline{B}_R(x_0, y_0) \subset D$. Moreover, by the upper semicontinuity of F and by Proposition 1.1.3 in [2], the set $F(\overline{B}_R(x_0, y_0))$ is compact, hence there exists $M > 0$ such that

$$\sup\{||v||;\ v \in F(x, y);\ (x, y) \in \overline{B}_R(x_0, y_0)\} \leq M < +\infty.$$

Let ϕ_V the continuous function appearing in Definition 2.2.

Since $V(.)$ is continuous on $D(V)$ (e.g. [9]), by possibly decreasing R one can assume that for all $y \in B_R(y_0) \cap D(V)$

$$|V(y) - V(y_0)| \leq 1.$$

Put

$$S := \sup\{\phi_v(y_1, y_2, z_1, z_2); y_i \in \overline{B}_r(y_0), z_i \in [V(y_0) - 1, V(y_0) + 1], i = 1, 2\},$$

By Hypothesis 2.3 iii) there exists $T > 0$ such that

$$\max\left\{\int_0^T (p(t) + M)\mathrm{d}t, T\left(||y_0|| + 2\int_0^T (p(t) + M)\mathrm{d}t\right)\right\} < \frac{R}{2}.$$

We shall prove the existence of solution of the problem (1.1) on the interval $[0, T]$.

For each $m \geq 1$ and $1 \leq j \leq m$ we define

$$t_m^j = \frac{jT}{m},\ I_m^j = [t_m^{j-1}, t_m^j],\ x_m^0 = x_0,\ y_m^0 = y_0,$$

and for $t \in I_m^j$ we define

$$x_m(t) = x_m^j + (t - t_m^j)y_m^j + \int_{t_m^j}^t (t - s)[f(s, x_m^j, y_m^j) + u_m^j]\mathrm{d}s, \tag{3.1}$$

where $u_m^j \in F(x_m^j, y_m^j)$, $j = 0, 1, ..., m-1$,

$$x_m^{j+1} = x_m^j + \frac{T}{m} y_m^j + \int_{t_m^j}^{t_m^{j+1}} (t_m^{j+1} - s)[f(s, x_m^j, y_m^j) + u_m^j] \mathrm{d}s, \tag{3.2}$$

$$y_m^{j+1} = y_m^j + \int_{t_m^j}^{t_m^{j+1}} [f(s, x_m^j, y_m^j) + u_m^j] \mathrm{d}s. \tag{3.3}$$

Obviously, from (3.1), if $t \in I_m^j$, we have

$$x'_m(t) = y_m^j + \int_{t_m^j}^{t} [f(s, x_m^j, y_m^j) + u_m^j] \mathrm{d}s, \tag{3.4}$$

$$x''_m(t) = f(t, x_m^j, y_m^j) + u_m^j. \tag{3.5}$$

For $t \in I_m^j$ we set $f_m(t) = f(t, x_m^j, y_m^j)$.

From (3.3), for any $j = 0, 1, ..., m-1$ one has

$$||y_m^j - y_0|| \leq \int_0^T (p(t) + M) \mathrm{d}t < R$$

and hence $||y_m^j|| \leq ||y_0|| + \int_0^T (p(t) + M) \mathrm{d}t$.

Therefore, from (3.4) and the choice of T, if $t \in I_m^j$

$$||x'_m(t) - y_0|| \leq ||y_m^j - y_0|| + \int_{t_m^j}^{t} [f(s, x_m^j, y_m^j) + u_m^j] \mathrm{d}s$$

$$\leq 2 \int_0^T (p(t) + M) \mathrm{d}t < R.$$

On the other hand, since

$$x_m^j = x_0 + \frac{T}{m} \sum_{k=0}^{j-1} y_m^k + \sum_{k=0}^{j} \int_{t_m^k}^{t_m^{k+1}} (t_m^{k+1} - s)[f(s, x_m^k, y_m^k) + u_m^k] \mathrm{d}s,$$

we get

$$||x_m^j - x_0|| \leq \frac{T}{m} \sum_{k=0}^{j-1} ||y_m^k|| + \sum_{k=0}^{j} \int_{t_m^k}^{t_m^{k+1}} |t_m^{k+1} - s|(p(s) + M) \mathrm{d}s$$

$$\leq \frac{T}{m} j \left(||y_0|| + \int_0^T (p(t) + M) \mathrm{d}t \right) + \int_0^{t_m^{j+1}} T(p(s) + M) \mathrm{d}s$$

$$\leq T||y_0|| + 2T \int_0^T (p(t) + M) \mathrm{d}t < R.$$

Therefore, from (3.1) and the choice of T, if $t \in I_m^j$

$$\begin{aligned}||x_m(t) - x_0|| &\leq ||x_m^j - x_0|| + (t - t_m^j)||y_m^j|| \\ &+ \int_{t_m^j}^t |t-s|(||f(s, x_m^j, y_m^j)|| + ||u_m^j||)\mathrm{d}s \\ &\leq T||y_0|| + 2T\int_0^T (p(t)+M)\mathrm{d}t + T(||y_0|| \\ &+ \int_0^T (p(t)+M)\mathrm{d}t + T\int_0^T (p(t)+M)\mathrm{d}t \\ &= 2T||y_0|| + 4T\int_0^T (p(t)+M)\mathrm{d}t < R.\end{aligned}$$

So from (3.1), (3.4) and (3.5) it follows that

$$||x_m''(t)|| \leq p(t) + M \quad \forall t \in [0,T], \tag{3.6}$$

$$||x_m'(t)|| \leq ||y_0|| + R \quad \forall t \in [0,T], \tag{3.7}$$

$$||x_m(t)|| \leq ||x_0|| + R \quad \forall t \in [0,T]. \tag{3.8}$$

At the same time, since for all $t \in I_m^j$

$$||x_m'(t) - y_m^j|| \leq \int_{t_m^j}^{t_m^{j+1}} (p(t)+M)\mathrm{d}t,$$

$$||x_m(t) - x_m^j|| \leq \frac{T}{m}\left(||y_0|| + \int_0^T (p(t)+M)\mathrm{d}t\right) + \frac{T}{m}\int_{t_m^j}^{t_m^{j+1}} (p(t)+M)\mathrm{d}t,$$

using the absolute continuity of the Lebesgue integral we infer that for all $t \in [0,T]$

$$(x_m(t), x_m'(t), x_m''(t) - f_m(t)) \in graphF + \epsilon(m)(B \times B \times B), \tag{3.9}$$

where $\epsilon(m) \to 0$ as $m \to \infty$.

By (3.6)–(3.8) we obtain that $x_m''(.)$ is bounded in $L^2([0,T], R^n)$ and $x_m(.)$, $x_m'(.)$ are bounded in $C([0,T], R^n)$. Moreover, for all $t', t'' \in [0,T]$

$$||x_m(t') - x_m(t'')|| \leq \left|\int_{t'}^{t''} ||x_m'(s)||\mathrm{d}s\right| \leq (||y_0|| + R)|t' - t''|,$$

$$||x_m'(t') - x_m'(t'')|| \leq \left|\int_{t'}^{t''} ||x_m''(s)||\mathrm{d}s\right| \leq \left|\int_{t'}^{t''} (p(s)+M)\mathrm{d}s\right|,$$

i.e. the sequence $x_m(.)$ is equi lipschitzian and the sequence $x_m'(.)$ is equi uniformly continuous.

Applying Theorem 0.3.4 in [2] we deduce the existence of a subsequence (again denoted by) $x_m(.)$ and an absolutely continuous function $x(.) : [0,T] \to$

R^n such that $x_m(.)$ converges uniformly to $x(.)$, $x'_m(.)$ converges uniformly to $x'(.)$ and $x''_m(.)$ converges weakly in $L^2([0,T],R^n)$ to $x''(.)$.

From Hypothesis 2.1 and Theorem 1.4.1 in [2] we find that

$$x''(t)-f(t,x(t),x'(t))\in coF(x(t),x'(t))\subset \partial_F V(x'(t)) \quad a.e.\ [0,T].$$

Since the mappings $x'(.)$ is absolutely continuous and $x''(t)\in\partial_F V(x'(t))$ a.e. $([0,T])$ we apply Theorem 2.2 in [5] and we deduce that there exists $T_1>0$ such that the mapping $t\to V(x'(t))$ is absolutely continuous on $[0,\min\{T,T_1\}]$ and

$$(V(x'(t)))'=<x''(t),x''(t)-f(t,x(t),x'(t))> \quad a.e.\ [0,\min\{T,T_1\}].$$

Without loss of generality we may assume that $T=\min\{T,T_1\}$.

Therefore

$$\begin{aligned} V(x'(T))-V(y_0)=&\int_0^T ||x''(t)||^2\mathrm{d}t \\ &-\int_0^T <x''(s),f(s,x(s),x'(s))>\mathrm{d}s. \end{aligned} \tag{3.10}$$

Since

$$x''_m(t)-f_m(t)=u_m^j\in F(x_m^j,y_m^j)\subset\partial_F V(y_m^j),\quad t\in I_m^j,$$

using the properties of the mapping $V(.)$ and the definition of S we have for $j=1,2,...,m$ and m fixed

$$\begin{aligned} V(x'_m(t_m^{j+1}))-V(x'_m(t_m^j)) &\geq <u_m^j,x'_m(t_m^{j+1})-x'_m(t_m^j)> \\ &-\phi_V(x'_m(t_m^{j+1}),x'_m(t_m^j),V(x'_m(t_m^{j+1})),V(x'_m(t_m^j))) \\ &\qquad (1+||u_m^j||^2)||x'_m(t_m^{j+1})-x'_m(t_m^j)||^2 \\ &\geq <u_m^j,x'_m(t_m^{j+1})-x'_m(t_m^j)> \\ &-S(1+||u_m^j||^2)\left|\left|\int_{t_m^j}^{t_m^{j+1}} x''_m(s)\mathrm{d}s\right|\right|^2 \\ &\geq \int_{t_m^j}^{t_m^{j+1}}||x''_m(s)||^2\mathrm{d}s-\int_{t_m^j}^{t_m^{j+1}}<f_m(s),x''_m(s)>\mathrm{d}s \\ &-S(1+M^2)\left|\left|\int_{t_m^j}^{t_m^{j+1}} x''_m(s)\mathrm{d}s\right|\right|^2. \end{aligned}$$

By adding the last inequality for $j = 1, 2, ..., m$, we obtain

$$
\begin{aligned}
V(x'_m(T)) - V(y_0) &\geq \int_0^T ||x''_m(t)||^2 \mathrm{d}t - \int_0^T < f_m(t), x''_m(t) > \mathrm{d}t \\
&- S(1+M^2)\frac{T}{m}||x'_m||^2_{L^2} \\
&\geq \int_0^T ||x''_m(t)||^2 \mathrm{d}t - \int_0^T < f_m(t), x''_m(t) > \mathrm{d}t \\
&- S(1+M^2)\frac{T}{m}\int_0^T (p(t)+M)^2 \mathrm{d}t.
\end{aligned}
\tag{3.11}
$$

The convergence of $f_m(.)$ in $L^2([0,T], R^n)$ and the convergence of $x''_m(.)$ in the weak topology of $L^2([0,T], R^n)$ implies that

$$\lim_{m\to\infty} \int_0^T < f_m(t), x''_m(t) > \mathrm{d}t = \int_0^T < f(t, x(t), x'(t)), x''(t) > \mathrm{d}t.$$

Passing to the limit with $m \to \infty$ in (3.11) we get

$$
\begin{aligned}
V(x'(T)) - V(y_0) \geq \limsup_{m\to\infty} &\int_0^T ||x''_m(t)||^2 \mathrm{d}t \\
&- \int_0^T < f(t, x(t), x'(t)), x''(t) > \mathrm{d}t.
\end{aligned}
$$

So, from (3.10) it follows that

$$\limsup_{m\to\infty} \int_0^T ||x'_m(t)||^2 \mathrm{d}t \leq \int_0^T ||x''(t)||^2 \mathrm{d}t$$

and, since $\{x''_m(.)\}_m$ converges weakly in $L^2([0,T], R^n)$ to $x''(.)$, by the lower semicontinuity of the norm in $L^2([0,T], R^n)$ (e.g. Prop. III.30 in [3]) we obtain that

$$\lim_{m\to\infty} \int_0^T ||x''_m(t)||^2 \mathrm{d}t = \int_0^T ||x''(t)||^2 \mathrm{d}t\,,$$

i.e., $\{x''_m(.)\}$ converges strongly in $L^2([0,T], R^n)$. Hence, there exists a subsequence (still denoted) $x''_m(.)$ that converges pointwise to $x''(.)$. Since, by Hypothesis 2.3, *graph*(F) is closed (e.g. [2], p. 41) from (3.9) we infer that

$$d((x(t), x'(t), x''(t) - f(t, x(t), x'(t))), graph(F)) = 0 \quad a.e. \ [0,T].$$

Thus

$$x''(t) \in F(x(t), x'(t)) + f(t, x(t), x'(t)), \quad a.e. \ [0,T].$$

Obviously, $x(.)$ satisfies the initial conditions and the proof is complete. □

REMARK 3.2. If $V(.) : R^n \to R$ is a proper lower semicontinuous convex function then (e.g. [9]) $\partial_F V(x) = \partial V(x)$, where $\partial V(.)$ is the subdifferential in the sense of convex analysis of $V(.)$, and Theorem 3.1 yields the result in [7]. If $f(t, x, y) \equiv 0$ then Theorem 3.1 yields the result in [6], which contains as a particular case (when $V(.)$ is convex) the result in [11].

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Faculty of Mathematics and Informatics
University of Bucharest
Str. Academiei 14
RO-010014 Bucharest, Romania
E-mail: acernea@math.math.unibuc.ro