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ON A CONJECTURE OF LIVINGSTON

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Abstract. Let D denote the open unit disc and $f : D \to \overline{\mathbf{C}}$ be meromorphic and injective in D. We further assume that f has a simple pole in the point $p \in (0,1)$ and an expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n, \quad |z| < p.$$

Especially, we consider f that map D onto a domain whose complement with respect to $\overline{\mathbf{C}}$ is convex. Concerning a (sharper) conjecture of Livingston ([5]) we prove that for $n \geq 2$ the inequality

$$\mathrm{Re}\,(a_n(f)) \geq \frac{1 + p^{2n}}{p^{n-1}(1+p)^2}$$

is valid.

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In the last century, many beautiful results have been proved in Geometric Function Theory for functions holomorphic in the open unit disc D that map D conformally onto a convex domain.

The present paper is devoted to a pendant of the family of convex functions, the family of concave univalent functions with pole $p \in (0,1)$ denoted by $Co(p)$ here. To be precise, we say that a function $f : D \to \overline{\mathbf{C}}$ belongs to the family $Co(p)$ if and only if:

(1) f is meromorphic in D and has a simple pole in the point $p \in (0,1)$.

(2) f has an expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n, \quad |z| < p.$$

(3) f maps D conformally onto a set whose complement with respect to $\overline{\mathbf{C}}$ is convex.

There are results on $Co(p)$ that resemble very much those on convex functions, for example it has been proved in [3] that $|a_n(f)| > 1$ for $f \in Co(p)$. Other results look very different from the analogous results on convex functions. Results of this type are the exact domains of variability of the Taylor coefficients $a_n(f)$, $f \in Co(p)$. J. Miller proved in [6] in principle that the inequality

$$\left| a_2(f) - \frac{1+p^2+p^4}{p(1+p^2)} \right| \le \frac{p}{1+p^2} \tag{1}$$

describes the exact domain of variability of $a_2(f), f \in Co(p)$ (compare [5], [1], [3], too).
Livingston proved in [5] that the lower bound in

$$\operatorname{Re}(a_3(f)) \ge \frac{1-p^2+p^4}{p^2}, \; f \in Co(p), \tag{2}$$

is sharp for any $p \in (0,1)$. The functions for which the bounds are attained in (1) and (2) map D onto the whole extended plane minus a line segment. The consideration of these extremal functions lead Livingston in [5] to the conjecture

$$\operatorname{Re}(a_n(f)) \ge \frac{1+p^{2n}}{p^{n-1}(1+p^2)}, \; f \in Co(p), n \ge 2, \; p \in (0,1). \tag{3}$$

In the present article we prove the existence of a positive lower bound for $\operatorname{Re}(a_n(f)), f \in Co(p), \; n \ge 2, \; p \in (0,1)$, which differs from the conjectured bound in (3) by the factor

$$\frac{1+p^2}{(1+p)^2} \in \left(\frac{1}{2}, 1\right).$$

This will be the content of Theorem 2. As a preparation for its proof we show

Theorem 1. *Let $p \in (0,1), f \in Co(p)$ and $c \in \overline{\mathbf{C}} \setminus f(D)$. Then the sharp inequalities*

$$-\frac{p}{(1-p)^2} \le \operatorname{Re}(c) \le -\frac{p}{(1+p)^2} \tag{4}$$

are valid. Equality in (4) *is attained if and only if*

$$f_e(z) = \frac{z}{\left(1-\frac{z}{p}\right)(1-zp)}. \tag{5}$$

and $c = f_e(1)$ in the left inequality (4)*, resp. $c = f_e(-1)$ in the right inequality* (4).

Proof. Since $\overline{\mathbf{C}} \setminus f(D)$ is starlike with respect to c and f is normalized as defined above and has a simple pole in the point p, the function

$$F(z) := \frac{(1-\frac{z}{p})(1-zp)f'(z)}{f(z)-c},$$

resp. its holomorphic extension from $D \setminus \{p\}$ onto D has the following properties
(1) $\operatorname{Re}(F(z)) > 0$ for $z \in D$ (compare [5], Theorem 6).
(2) $F(0) = -\frac{1}{c}$ and $F(p) = \frac{1-p^2}{p}$.

Let $c = x + \mathrm{i}y$. From the properties of the function F we conclude that $x < 0$ and that there exists a function φ holomorphic in D such that $\varphi(D) \subset D$, $\varphi(0) = 0$ and

$$-\frac{F(z)(x^2+y^2) \,-\, \mathrm{i}y}{x} = \frac{1-\varphi(z)}{1+\varphi(z)},\ z \in D.$$

Hence, there exists a function Φ holomorphic in D such that $\Phi(D) \subset \overline{D}$,

$$-\frac{F(z)(x^2+y^2) \,-\, \mathrm{i}y}{x} = \frac{1-z\Phi(z)}{1+z\Phi(z)},\ z \in D,$$

and

$$-\frac{\frac{1-p^2}{p}(x^2+y^2) \,-\, \mathrm{i}y}{x} = \frac{1-p\Phi(p)}{1+p\Phi(p)}.$$

This equation together with $\Phi(D) \subset \overline{D}$ yields that for every $c = x + \mathrm{i}y \in \overline{\mathbf{C}} \setminus f(D)$ there exists a $w \in \overline{D}$ such that

$$-\frac{\frac{1-p^2}{p}(x^2+y^2) \,-\, \mathrm{i}y}{x} = \frac{1-pw}{1+pw} =: u + \mathrm{i}v, \tag{6}$$

where $u + \mathrm{i}v$ varies in the disc described by

$$\left(u - \frac{1+p^2}{1-p^2}\right)^2 + v^2 \le \left(\frac{2p}{1-p^2}\right)^2. \tag{7}$$

From (6) we get

$$y = x\,v$$

and therefore

$$x = -\frac{u\,p}{(1-p^2)(1+v^2)}.$$

According to (7), this implies (4), where equality in the left inequality is attained only for

$$u = \frac{1+p}{1-p},\ v = 0, \tag{8}$$

and in the right inequality only for

$$u = \frac{1-p}{1+p},\ v = 0. \tag{9}$$

The formula (8) means that $\Phi(p) = -1$. According to the maximum principle this implies $\Phi \equiv -1$. The initial value problem

$$\frac{p}{(1-p)^2}\,\frac{(1-\frac{z}{p})(1-z\,p)f'(z)}{f(z) + \frac{p}{(1-p)^2}} = \frac{1+z}{1-z},\ f(0) = 0,$$

has as its unique solution the extremal function f_e defined in (6), which maps D onto

$$\overline{\mathbf{C}} \setminus \left[-\frac{p}{(1-p)^2}, -\frac{p}{(1+p)^2}\right]$$

(compare [5], [6] and[1]). The reasoning concerning the right inequality in (4) is analogous with (9) and $\Phi(p) = 1$. □

THEOREM 2. *Let $p \in (0,1)$, $f \in Co(p)$ and $n \geq 2$. Then the inequality*

$$\mathrm{Re}(a_n(f)) \geq \frac{1+p^{2n}}{p^{n-1}(1+p)^2} \tag{10}$$

is valid.

Proof. In principal, we proceed as in [1], where the inequality

$$|a_n(f)| \geq \frac{1+p^{2n}}{p^{n-1}(1+p)^2}$$

was proved and we use some arguments from [2] where the Taylor coefficients of meromorphic univalent functions have been considered. For $n \geq 2$ the function h defined by

$$h(z) := \begin{cases} \left(1 - z^n\left(p^n + \frac{1}{p^n}\right) + z^{2n}\right) f(z), & z \in D \setminus \{p\}, \\ \lim_{0<|w-p|\to 0} h(w), & z = p \end{cases}$$

is bounded and holomorphic in D. Therefore the angular limits $h(\mathrm{e}^{\mathrm{i}\theta})$, and, in turn, the angular limits $f(\mathrm{e}^{\mathrm{i}\theta})$ exist almost everywhere in $[0, 2\pi)$ by Fatou's theorem (see [4], chapter IX). Apparently,

$$a_n(h) = a_n(f).$$

As a consequence of a theorem of F. Riesz (see for instance [4], p. 404) we get

$$\lim_{R\to 1-0} \int_0^{2\pi} \left| h(R\mathrm{e}^{\mathrm{i}\theta}) - h(\mathrm{e}^{\mathrm{i}\theta}) \right| \mathrm{d}\theta = 0.$$

This together with the residue theorem and the above yields

$$a_n(f) = a_n(h) = \frac{1}{2\pi} \int_0^{2\pi} h(\mathrm{e}^{\mathrm{i}\theta}) \mathrm{e}^{-\mathrm{i}n\theta} \mathrm{d}\theta$$

$$= -\frac{1}{2\pi} \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta}) \left(p^n + \frac{1}{p^n} - 2\cos(n\theta)\right) \mathrm{d}\theta.$$

Now, we use the the right inequality in (4) for $c = f(\mathrm{e}^{\mathrm{i}\theta})$ to get immediately the inequality (10). □

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