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ON CERTAIN SUBCLASS OF PRESTARLIKE FUNCTIONS DEFINED BY SALAGEAN OPERATOR

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Abstract. The object of the present paper is to investigate several interesting properties of the class $T_n(\lambda, \alpha, \gamma)$ consisting of prestarlike functions with negative coefficients defined by Salagean operator. Coefficient estimates, distortion theorems, closure theorems and modified Hadamard products of several functions belonging to the class $T_n(\lambda, \alpha, \gamma)$ are determined. Also radii of close-to-convexity, starlikeness and convexity are determined. Also we obtain integral operators for this class.
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1. INTRODUCTION

Let A denote the class of (normalized) functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc

$$U = \{z : z \in C \text{ and } |z| < 1\},$$

and let S denote the subclass of A consisting of functions which are also univalent in U. Then a function $f(z)$ in S is said to be starlike of order α if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U;\ 0 \leq \alpha < 1). \tag{1.2}$$

We denote by $S^*(\alpha)$ the class of all functions in S which are starlike of order α in U. It is well-known that

$$S^*(\alpha) \subseteq S^*(0) = S^*.$$

The class $S^*(\alpha)$, introduced by Robertson [9], was studied subsequently by Schild [11], MacGregor [6] and Pinchuk [8]. Moreover, the function:

$$s_\gamma(z) = \frac{z}{(1-z)^{2(1-\gamma)}} \tag{1.3}$$

is the familiar extremal function for the class $S^*(\gamma)$. Setting

$$c(\gamma, k) = \frac{\prod_{i=2}^{k}(i - 2\gamma)}{(k-1)!} \quad (k \geq 2), \tag{1.4}$$

$s_\gamma(z)$ can be written in the form:

$$s_\gamma(z) = z + \sum_{k=2}^{\infty} c(\gamma, k) z^k. \tag{1.5}$$

We note that $c(\gamma, k)$ is a decreasing function in γ and that

$$\lim_{k\to\infty} c(\gamma, k) = \begin{cases} \infty & (\gamma < 1/2) \\ 1 & (\gamma = 1/2) \\ 0 & (\gamma > 1/2). \end{cases} \tag{1.6}$$

For a function $f(z)$ in S, we define

$$D^0 f(z) = f(z), \tag{1.7}$$

$$D^1 f(z) = Df(z) = zf'(z), \tag{1.8}$$

and

$$D^n f(z) = D\left(D^{n-1} f(z)\right) \quad (n \in N = \{1, 2, \cdots\}). \tag{1.9}$$

The differential operator D^n was introduced by Salagean [10].

For α $(0 \le \alpha < 1)$ and λ $(0 \le \lambda < 1)$, we say that a function $f(z) \in A$ is in the class $S(\lambda, \alpha)$ if and only if

$$\operatorname{Re}\left\{ \frac{\dfrac{zf'(z)}{f(z)}}{\lambda \dfrac{zf'(z)}{f(z)} + (1-\lambda)} \right\} > \alpha \quad (z \in U). \tag{1.10}$$

Let $(f * g)(z)$ denote the Hadamard product or convolution of two functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.11}$$

then

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.12}$$

Let $S_n(\lambda, \alpha, \gamma)$ be the subclass of A consisting of functions $f(z)$ such that

$$\operatorname{Re}\left\{ \frac{\dfrac{z\varphi'_{n,\gamma}(z)}{\varphi_{n,\gamma}(z)}}{\lambda \dfrac{z\varphi'_{n,\gamma}(z)}{\varphi_{n,\gamma}(z)} + (1-\lambda)} \right\} > \alpha \quad (z \in U). \tag{1.13}$$

where $0 \le \alpha < 1$, $0 \le \lambda < 1$, $0 \le \gamma < 1$, $n \in N_0 = N \cup \{0)\}$ and

$$\varphi_{n,\gamma}(z) = \left(D^n f * s_\gamma\right)(z). \tag{1.14}$$

We observe that $S_0(0,\alpha,\gamma) = R_\gamma(\alpha)$ is the class of γ-prestarlike functions of order α, which was introduced by Sheil-Small et al. [13].

Let T denote the subclass of A consisting of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0). \tag{1.15}$$

We denote by $T(\lambda,\alpha)$ and $T_n(\lambda,\alpha,\gamma)$ the classes obtained by taking intersections, respectively, of the classes $S(\lambda,\alpha)$ and $S_n(\lambda,\alpha,\gamma)$ with the class T. The class $T(\lambda,\alpha)$ was studied by Altintas and Owa [1].

We note that, by specializing the parameters λ, α, γ and n, we obtain the following subclasses studied by various authors:

(i) $T_n(0,\alpha,\gamma) = R[\gamma,\alpha,n]$ (Aouf and Salagean [4]);

(ii) $T_0(0,\gamma,\gamma) = R_\gamma^*$ (Silverman and Silvia [15]);

(iii) $T_0(0,\alpha,\gamma) = R_\gamma[\alpha]$ (Silverman and Silvia [16], Uralegaddi and Sarangi [17] and Aouf and Salagean [3]);

(iv) $T_1(0,\alpha,\gamma) = C_\gamma[\alpha]$ (Owa and Uralegaddi [7] and Aouf and Salagean [3]);

(v) $T_n(\lambda,\alpha,1/2) = T_n(\lambda,\alpha)$ (Aouf and Cho [2]);

(vi) $T_n(0,\alpha,1/2) = T(n,\alpha)$ (Hurand Oh [5]);

(vii) $T_0(0,\alpha,1/2) = T'(\alpha)$ and $T_1(0,\alpha,1/2) = C(\alpha)$ (Silverman [14]);

(viii) $T_0(\lambda,\alpha,1/2) = T(\lambda,\alpha)$ and $T_1(\lambda,\alpha,1/2) = C(\lambda,\alpha)$ (Altintas and Owa [1]).

In the present paper we purpose to investigate several important properties and characteristics of the class $T_n(\lambda,\alpha,\gamma)$ which we have defined here.

2. COEFFICIENT ESTIMATES

THEOREM 1. *Let the function $f(z)$ be defined by* (1.15). *Then $f(z) \in T_n(\lambda,\alpha,\gamma)$ if and only if*

$$\sum_{k=2}^{\infty} k^n \{k - \alpha[1 + \lambda(k-1)]\} c(\gamma,k) a_k \leq (1-\alpha). \tag{2.1}$$

The result is sharp.

Proof. It is known from [1] that a necessary and sufficient condition for $f(z)$ defined by (1.15) to be in the class $T_0(\lambda,\alpha) = T(\lambda,\alpha)$ is that

$$\sum_{k=2}^{\infty} \{k - \alpha[1 + \lambda(k-1)]\} a_k \leq (1-\alpha).$$

Since

$$\left(D^n f * s_\gamma\right)(z) = z - \sum_{k=2}^{\infty} k^n c(\gamma,k) a_k z^k, \quad (a_k \geq 0)$$

where $s_\gamma(z)$ is given by (1.5), the result (2.1) follows. Further, we can see that the function $f(z)$ given by

$$f(z) = z - \frac{(1-\alpha)}{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)} z^k \quad (k \geq 2) \tag{2.2}$$

is the extremal function of Theorem 1.

Corollary 1. *Let the function $f(z)$ defined by* (1.15) *be in the class $T_n(\lambda,\alpha,\gamma)$. Then we have*

$$a_k \leq \frac{(1-\alpha)}{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)} \quad (k \geq 2). \tag{2.3}$$

The equality in (2.3) *is attained for the function $f(z)$ given by* (2.2).

3. SOME PROPERTIES OF THE CLASS $T_n(\lambda,\alpha,\gamma)$

Theorem 2. *Let $0 \leq \alpha < 1$, $0 \leq \lambda_1 \leq \lambda_2 < 1$, $0 \leq \gamma < 1$ and $n \in N_0$. Then*

$$T_n(\lambda_1,\alpha,\gamma) \subseteq T_n(\lambda_2,\alpha,\gamma).$$

Proof. It follows from Theorem 1 that

$$\sum_{k=2}^{\infty} k^n\{k-\alpha[1+\lambda_2(k-1)]\}c(\gamma,k)a_k \leq$$

$$\sum_{k=2}^{\infty} k^n\{k-\alpha[1+\lambda_1(k-1)]\}c(\gamma,k)a_k \leq (1-\alpha)$$

for $f(z) \in T_n(\lambda_1,\alpha,\gamma)$. Hence $f(z) \in T_n(\lambda_2,\alpha,\gamma)$.

Theorem 3. *Let $0 \leq \alpha < 1$, $0 \leq \lambda < 1$, $0 \leq \gamma < 1$ and $n \in N_0$. Then*

$$T_{n+1}(\lambda_1,\alpha,\gamma) \subseteq T_n(\lambda,\alpha,\gamma).$$

The proof follows immediately from Theorem 1.

4. DISTORTION THEOREMS

Theorem 4. *Let the function $f(z)$ defined by* (1.15) *be in the class $T_n(\lambda,\alpha,\gamma)$, with $0 \leq \alpha < 1$, $0 \leq \lambda < 1$, $0 \leq \gamma \leq 1/2$ and $n \in N_0$. Then*

$$|D^i f(z)| \geq |z| - \frac{(1-\alpha)}{2^{n+1-i}[2-\alpha(1+\lambda)](1-\gamma)}|z|^2 \tag{4.1}$$

and

$$|D^i f(z)| \leq |z| + \frac{(1-\alpha)}{2^{n+1-i}[2-\alpha(1+\lambda)](1-\gamma)}|z|^2, \tag{4.2}$$

for $z \in U$, *where* $0 \le i \le n$. *The equalities in* (4.1) *and* (4.2) *are attained for the function f(z) given by*

$$D^i f(z) = z - \frac{(1-\alpha)}{2^{n+1-i}[2-\alpha(1+\lambda)](1-\gamma)} z^2. \tag{4.3}$$

Proof. Note that $f(z) \in T_n(\lambda, \alpha, \gamma)$ if and only if $D^i f(z) \in T_{n-i}(\lambda, \alpha, \gamma)$ and that

$$D^i f(z) = z - \sum_{k=2}^{\infty} k^i a_k z^k. \tag{4.4}$$

Since $k^n\{k - \alpha[1 + \lambda(k-1)]\}$ and $c(\gamma, k)$, $0 \le \gamma \le 1/2$ are increasing functions of k $(k \ge 2)$ for a fixed n, since $f(z) \in T_n(\lambda, \alpha, \gamma)$, in view of Theorem 1, we have

$$2^{n-i}[2-\alpha(1+\lambda)]c(\gamma,2) \sum_{k=2}^{\infty} k^i a_k \le \tag{4.5}$$

$$\sum_{k=2}^{\infty} k^n \{k - \alpha[1+\lambda(k-1)]\} c(\gamma, k) a_k \le (1-\alpha),$$

that is, that

$$\sum_{k=2}^{\infty} k^i a_k \le \frac{(1-\alpha)}{2^{n+1-i}[2-\alpha(1+\lambda)](1-\gamma)}. \tag{4.6}$$

It follows from (4.4) and (4.6) that

$$|D^i f(z)| \ge |z| - |z|^2 \sum_{k=2}^{\infty} k^i a_k \tag{4.7}$$

$$\ge |z| - \frac{(1-\alpha)}{2^{n+1-i}[2-\alpha(1+\lambda)](1-\gamma)} |z|^2$$

and

$$|D^i f(z)| \le |z| + |z|^2 \sum_{k=2}^{\infty} k^i a_k \tag{4.8}$$

$$\le |z| + \frac{(1-\alpha)}{2^{n+1-i}[2-\alpha(1+\lambda)](1-\gamma)} |z|^2$$

Finally, we can see that the results in Theorem 4 are attained for the function $f(z)$ given by (4.3).

Corollary 2. *Let the function* $f(z)$ *defined by* (1.15) *be in the class* $T_n(\lambda, \alpha, \gamma)$. *Then we have*

$$|f(z)| \ge |z| - \frac{(1-\alpha)}{2^{n+1}[2-\alpha(1+\lambda)](1-\gamma)} |z|^2 \tag{4.9}$$

and

$$|f(z)| \leq |z| + \frac{(1-\alpha)}{2^{n+1}[2-\alpha(1+\lambda)](1-\gamma)}|z|^2, \tag{4.10}$$

for $z \in U$. *The equalities in* (4.9) *and* (4.10) *are attained for the function* $f(z)$ *given by*

$$f(z) = z - \frac{(1-\alpha)}{2^{n+1}[2-\alpha(1+\lambda)](1-\gamma)}z^2. \tag{4.11}$$

Proof. Taking $i = 0$ in Theorem 4, we can easily show (4.9) and (4.10).

Corollary 3. *Let the function* $f(z)$ *defined by* (1.15) *be in the class* $T_n(\lambda, \alpha, \gamma)$. *Then we have*

$$|f'(z)| \geq 1 - \frac{(1-\alpha)}{2^{n}[2-\alpha(1+\lambda)](1-\gamma)}|z| \tag{4.12}$$

and

$$|f'(z)| \leq 1 + \frac{(1-\alpha)}{2^{n+1}[2-\alpha(1+\lambda)](1-\gamma)}|z|, \tag{4.13}$$

for $z \in U$. *The equalities in* (4.12) *and* (4.13) *are attained for the function* $f(z)$ *given by* (4.11).

Proof. Note that $Df(z) = zf'(z)$. Hence taking $i = 1$ in Theorem 4, we have the next corollary.

Corollary 4. *Let the function* $f(z)$ *defined by* (1.15) *be in the class* $T_n(\lambda, \alpha, \gamma)$. *Then the unit disc* U *is mapped onto a domain that contains the disc*

$$|w| < \frac{2^{n+1}[2-\alpha(1+\lambda)](1-\gamma) - (1-\alpha)}{2^{n+1}[2-\alpha(1+\lambda)](1-\gamma)}. \tag{4.14}$$

The result is sharp with the extremal function $f(z)$ *given by* (4.11).

5. CLOSURE THEOREMS

Theorem 5. *The class* $T_n(\lambda, \alpha, \gamma)$ *is closed under linear combination.*

Proof. Let each of the functions $f_1(z)$ and $f_2(z)$ given by

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geq 0; \;\; j = 1,\ 2) \tag{5.1}$$

be in the class $T_n(\lambda, \alpha, \gamma)$. Then it is sufficient to show that the function $h(z)$ defined by

$$h(z) = tf_1(z) + (1-t)f_2(z) \quad (0 \leq t \leq 1) \tag{5.2}$$

is also in the class $T_n(\lambda,\alpha,\gamma)$. Since, for $0 \le t \le 1$,

$$h(z) = z - \sum_{k=2}^{\infty} \{t\, a_{k,1} + (1-t)\, a_{k,2}\}\, z^k, \tag{5.3}$$

with the aid of Theorem 1, we have

$$\sum_{k=2}^{\infty} \{k-\alpha[1+\lambda(k-1)]\} c(\gamma,k) \{t a_{k,1} + (1-t) a_{k,2}\} \le (1-\alpha) \quad (0 \le t \le 1) \tag{5.4}$$

which implies that $h(z) \in T_n(\lambda,\alpha,\gamma)$.

As a consequence of Theorem 5, there exist the extreme points of the class $T_n(\lambda,\alpha,\gamma)$.

Theorem 6. *Let*

$$f_1(z) = z \tag{5.5}$$

and

$$f_k(z) = z - \frac{(1-\alpha)}{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)} z^k \quad (k \ge 2). \tag{5.6}$$

Then $f(z)$ is in the class $T_n(\lambda,\alpha,\gamma)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} t_k f_k(z)\,, \tag{5.7}$$

where $t_k \ge 0$ $(k \ge 1)$ and $\sum_{k=1}^{\infty} = 1$.

Proof. Assume that

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} t_k f_k(z) \\ f(z) &= z - \sum_{k=2}^{\infty} \frac{(1-\alpha)}{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)} t_k z^k \end{aligned} \tag{5.8}$$

Then it follows that
(5.9)

$$\sum_{k=2}^{\infty} \frac{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)}{(1-\alpha)} \cdot \sum_{k=2}^{\infty} \frac{(1-\alpha)}{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)}\ t_k$$

$$= \sum_{k=2}^{\infty} t_k = 1 - t_1 \le 1.$$

Therefore, by Theorem 1, $f(z) \in T_n(\lambda,\alpha,\gamma)$.

Conversely, assume that the function $f(z)$ defined by (1.15) belongs to the class $T_n(\lambda, \alpha, \gamma)$. Then we have

$$a_k \leq \frac{(1-\alpha)}{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)} \quad (k \geq 2). \tag{5.10}$$

Setting

$$t_k = \frac{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)}{(1-\alpha)}\, a_k \quad (k \geq 2) \tag{5.11}$$

and

$$t_1 = 1 - \sum_{k=2}^{\infty} t_k, \tag{5.12}$$

we see that $f(z)$ can be expressed in the form (5.7). This completes the proof of Theorem 6.

Corollary 5. *The extreme points of the class $T_n(\lambda, \alpha, \gamma)$ are the functions $f_k(z)$ $(k \geq 1)$ given by Theorem 6.*

6. RADII OF CLOSE-TO-CONVEXITY, STARLIKENESS AND CONVEXITY

Theorem 7. *Let the function $f(z)$ defined by* (1.15) *be in the class $T_n(\lambda, \alpha, \gamma)$ $(0 \leq \alpha < 1,\ 0 \leq \lambda < 1,\ 0 \leq \gamma \leq 1/2$ and $n \in N_0)$. Then $f(z)$ is close-to-convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_1^*$, where*

$$r_1^* = \inf_k \left[\frac{(1-\rho)k^{n-1}\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)}{(1-\alpha)}\right]^{\frac{1}{k-1}} \qquad (k \geq 2). \tag{6.1}$$

The result is sharp, with the extremal function $f(z)$ being given by (2.2).

Proof. It is sufficient to show that

$$|f'(z) - 1| \leq 1 - \rho \quad \text{for} \quad |z| < r_1^*.$$

Indeed, we have

$$|f'(z) - 1| \leq \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| \leq 1 - \rho$ if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\rho}\right) a_k |z|^{k-1} \leq 1. \tag{6.2}$$

But Theorem 1 confirms that

$$\sum_{k=2}^{\infty} \frac{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)}{(1-\alpha)} a_k \leq 1. \tag{6.3}$$

Hence (6.2) will be true if

$$\frac{k|z|^{k-1}}{(1-\rho)} \le \frac{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)}{(1-\alpha)} \tag{6.4}$$

or if

$$|z| \le \left[\frac{(1-\rho)k^{n-1}\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)}{(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{6.5}$$

The theorem follows easily from (6.5).

THEOREM 8. *Let the function* $f(z)$ *defined by* (1.15) *be in the class* $T_n(\lambda,\alpha,\gamma)$ *(*$0 \le \alpha < 1$, $0 \le \lambda < 1$, $0 \le \gamma \le 1/2$ *and* $n \in N_0$*). Then* $f(z)$ *is starlike of order* ρ $(0 \le \rho < 1)$ *in* $|z| < r_2^*$, *where*

$$r_2^* = \inf_k \left[\frac{(1-\rho)k^{n}\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{6.6}$$

The result is sharp, the extremal function $f(z)$ *being given by* (2.2).

Proof. We must show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le 1-\rho \quad \text{for} \quad |z| < r_2^*.$$

Indeed, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}a_k|z|^{k-1}}.$$

Thus $\left|\frac{zf'(z)}{f(z)} - 1\right| \le 1-\rho$ if

$$\sum_{k=2}^{\infty}\left(\frac{k-\rho}{1-\rho}\right)a_k|z|^{k-1} \le 1. \tag{6.7}$$

Hence, by using (6.3), (6.7) will be true if

$$\left(\frac{k-\rho}{1-\rho}\right)|z|^{k-1} \le \frac{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)}{(1-\alpha)} \quad (k \ge 2), \tag{6.8}$$

that is, if

$$|z| \le \left[\frac{(1-\rho)k^{n}\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{6.9}$$

Theorem 8 follows easily from (6.9).

COROLLARY 6. *Let the function* $f(z)$ *defined by* (1.15) *be in the class* $T_n(\lambda,\alpha,\gamma)$ *(*$0 \le \alpha < 1$, $0 \le \lambda < 1$, $0 \le \gamma \le 1/2$ *and* $n \in N_0$*). Then* $f(z)$ *is convex of order* ρ $(0 \le \rho < 1)$ *in* $|z| < r_3^*$, *where*

$$r_3^* = \inf_k \left[\frac{(1-\rho)k^{n-1}\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{6.10}$$

The result is sharp, the extremal function $f(z)$ being given by (2.2).

7. INTEGRAL OPERATORS

THEOREM 9. *Let the function $f(z)$ defined by* (1.15) *be in the class $T_n(\lambda,\alpha,\gamma)$, and let d be a real number such that $d > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{d+1}{z^d}\int_0^z t^{d-1}\, f(t)\,\mathrm{d}t \quad (d > -1) \tag{7.1}$$

also belongs to the class $T_n(\lambda,\alpha,\gamma)$.

Proof. From the representation (7.1) of $F(z)$ it follows that

$$F(z) = z - \sum_{k=2}^{\infty} b_k z^k,$$

where

$$b_k = \left(\frac{d+1}{d+k}\right) a_k.$$

Therefore, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)b_k \\
=\;& \sum_{k=2}^{\infty} k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)\left(\frac{d+1}{d+k}\right)a_k \\
\le\;& \sum_{k=2}^{\infty} k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)a_k \\
\le\;& (1-\alpha),
\end{aligned}$$

since $f(z) \in T_n(\lambda,\alpha,\gamma)$. Hence, by Theorem 1, $F(z) \in T_n(\lambda,\alpha,\gamma)$.

THEOREM 10. *Let the function $F(z) = z - \sum_{k=2}^{\infty} a_k z^k$ $(a_k \ge 0)$ be in the class $T_n(\lambda,\alpha,\gamma)$ and let d be a real number such that $d > -1$. Then the function $f(z)$ defined by* (7.1) *is univalent in $|z| < R^*$, where*

$$R^* = \inf_k \left[\frac{(d+1)k^{n-1}\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)}{(d+k)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{7.2}$$

The result is sharp.

Proof. From (7.1), we have

$$f(z) = \frac{z^{1-d}\left[z^d F(z)\right]'}{(d+1)} \quad (d \ge -1) \tag{7.3}$$

$$= z - \sum_{k=2}^{\infty}\left(\frac{d+k}{d+1}\right) a_k z^k. \tag{7.4}$$

In order to obtain the required result it suffices to show that $|f'(z)-1|<1$ in $|z|<R^*$. Now

$$|f'(z)-1| \leq \sum_{k=2}^{\infty} \frac{k(d+k)}{(d+1)} a_k |z|^{k-1}.$$

Thus $|f'(z)-1|<1$ if

$$\sum_{k=2}^{\infty} \frac{k(d+k)}{(d+1)} a_k |z|^{k-1} < 1. \tag{7.5}$$

Hence, by using (6.3), (7.5) will be satisfied if

$$\frac{k(d+k)|z|^{k-1}}{(d+1)} \leq \frac{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)}{(1-\alpha)} \quad (k \geq 2)$$

or if

$$|z| \leq \left[\frac{(d+1)k^{n-1}\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)}{(d+k)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{7.6}$$

Therefore $f(z)$ is univalent in $|z|<R^*$. Sharpness follows if we take

$$f(z) = z - \frac{(1-\alpha)(d+k)}{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)(d+1)} z^k \quad (k \geq 2). \tag{7.7}$$

8. MODIFIED HADAMARD PRODUCTS

Let the functions $f_j(z)$ $(j = 1,\ 2)$ be defined by (5.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = z - \sum_{k=2}^{\infty} a_{k,1}\, a_{k,2}\, z^k. \tag{8.1}$$

THEOREM 11. *Let the functions $f_j(z)$ $(j=1,\ 2)$ defined by (5.1) be in the class $T_n(\lambda,\alpha,\gamma)$ $(0 \leq \alpha < 1,\ 0 \leq \lambda < 1,\ 0 \leq \gamma \leq 1/2$ and $n \in N_0)$. Then $f_1 * f_2(z)$ belongs to the class $T_n(\lambda, \beta(n,\lambda,\alpha,\gamma),\gamma)$, where*

$$\beta(n,\lambda,\alpha,\gamma) = 1 - \frac{(1-\lambda)(1-\alpha)^2}{2^{n+1}\{2-\alpha(1+\lambda)\}^2(1-\gamma)-(1+\lambda)(1-\alpha)^2}. \tag{8.2}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [12], we need to find the largest $\beta = \beta(n,\lambda,\alpha,\gamma)$ such that

$$\sum_{k=2}^{\infty} \frac{k^n\{k-\beta[1+\lambda(k-1)]\}c(\gamma,k)}{(1-\beta)} a_{k,1} a_{k,2} \leq 1. \tag{8.3}$$

Since

$$\sum_{k=2}^{\infty} \frac{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)}{(1-\alpha)} a_{k,1} \leq 1 \tag{8.4}$$

and

$$\sum_{k=2}^{\infty} \frac{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)}{(1-\alpha)} a_{k,2} \leq 1 \tag{8.5}$$

by the Cauchy-Schwarz inequality we have

$$\sum_{k=2}^{\infty} \frac{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)}{(1-\alpha)} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{8.6}$$

Thus it is sufficient to show that

$$\begin{aligned} &\frac{k^n\{k-\beta[1+\lambda(k-1)]\}c(\gamma,k)}{(1-\beta)} a_{k,1}a_{k,2} \\ &\leq \frac{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)}{(1-\alpha)} \sqrt{a_{k,1}a_{k,2}} \quad (k \geq 2), \end{aligned} \tag{8.7}$$

that is that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(1-\beta)\{k-\alpha[1+\lambda(k-1)]\}}{(1-\alpha)\{k-\beta[1+\lambda(k-1)]\}}. \tag{8.8}$$

Note that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(1-\alpha)}{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)} \quad (k \geq 2). \tag{8.9}$$

Consequently, we need only to prove that

$$\frac{(1-\alpha)}{k^n\{k-\alpha[1+\lambda(k-1)]\}c(\gamma,k)} \leq \frac{(1-\beta)\{k-\alpha[1+\lambda(k-1)]\}}{(1-\alpha)\{k-\beta[1+\lambda(k-1)]\}}, \tag{8.10}$$

or equivalently, that

$$\beta \leq 1 - \frac{(k-1)(1-\lambda)(1-\alpha)^2}{k^n\{k-\alpha[1+\lambda(k-1)]\}^2 c(\gamma,k) - [1+\lambda(k-1)](1-\alpha)^2}. \tag{8.11}$$

Since

$$A(k) = 1 - \frac{(k-1)(1-\lambda)(1-\alpha)^2}{k^n\{k-\alpha[1+\lambda(k-1)])\}^2 c(\gamma,k) - [1+\lambda(k-1)](1-\alpha)^2} \tag{8.12}$$

is an increasing function of k $(k \geq 2)$, letting $k=2$ in (8.12) we obtain

$$\beta \leq A(2) = 1 - \frac{(1-\lambda)(1-\alpha)^2}{2^{n+1}\{2-\alpha[1+\lambda]\}^2(1-\gamma) - (1+\lambda)(1-\alpha)^2}, \tag{8.13}$$

which completes the proof of Theorem 11.

Finally, by taking the functions $f_j(z)$ given by

$$f_j(z) = z - \frac{(1-\alpha)}{2^{n+1}\{2-\alpha[1+\lambda]\}(1-\gamma)} z^2 \quad (j = 1,\ 2) \tag{8.14}$$

we can see that the result is sharp.

Corollary 7. *For $f_1(z)$ and $f_2(z)$ as in Theorem 11, we have*

$$h(z) = z - \sum_{k=2}^{\infty} \sqrt{a_{k,1}a_{k,2}} z^k \tag{8.15}$$

belongs to the class $T_n(\lambda, \alpha, \gamma)$ ($0 \le \alpha < 1$, $0 \le \lambda < 1$, $0 \le \gamma \le 1/2$ and $n \in N_0$).

The result follows from the Cauchy-Schwarz inequality (8.6). It is sharp for the same functions as in Theorem 11.

Theorem 12. *Let the function $f_1(z)$ defined by (5.1) be in the class $T_n(\lambda, \alpha, \gamma)$ ($0 \le \alpha < 1$, $0 \le \lambda < 1$, $0 \le \gamma \le 1/2$ and $n \in N_0$) and the function $f_2(z)$ defined by (5.1) be in the class $T_n(\lambda, \tau, \gamma)$ ($0 \le \tau < 1$, $0 \le \lambda < 1$, $0 \le \gamma \le 1/2$ and $n \in N_0$). Then $f_1 * f_2(z) \in T_n(\lambda, \eta(n, \lambda, \alpha, \tau, \gamma), \gamma)$, where*

$$\eta(n, \lambda, \alpha, \tau, \gamma) = \tag{8.16}$$

$$1 - \frac{(1-\lambda)(1-\alpha)(1-\tau)}{2^{n+1}\{2-\alpha(1+\lambda)\}\{2-\tau(1+\lambda)\}(1-\gamma) - (1+\lambda)(1-\alpha)(1-\tau)}.$$

The result is the best possible for the functions:

$$f_1(z) = z - \frac{(1-\alpha)}{2^{n+1}\{2-\alpha(1+\lambda)\}(1-\gamma)} z^2 \tag{8.17}$$

and

$$f_2(z) = z - \frac{(1-\tau)}{2^{n+1}\{2-\tau(1+\lambda)\}(1-\gamma)} z^2. \tag{8.18}$$

Proof. Proceeding as in the proof of Theorem 11, we get

$$\eta \le B(k) = \tag{8.19}$$

$1 - (k-1)(1-\lambda)(1-\alpha)(1-\tau)/\{k^n c(\gamma, k)\{k - \alpha[1+\lambda(k-1)]\}\{k - \tau[1+\lambda(k-1)]\} - [1+\lambda(k-1)](1-\alpha)(1-\tau)\}$ $(k \ge 2)$.

Since the function $B(k)$ is an increasing function of k $(k \ge 2)$, setting $k = 2$ in (8.19), we get

$$\eta \le B(2) = 1 - \tag{8.20}$$

$$\frac{(1-\lambda)(1-\alpha)(1-\tau)}{2^{n+1}(1-\gamma)\{2-\alpha(1+\lambda)\}\{2-\tau(1+\lambda)\} - (1+\lambda)(1-\alpha)(1-\tau)}.$$

This completes the proof of Theorem 12.

Corollary 8. *Let the functions $f_j(z)$ ($j = 1, 2, 3$) defined by (5.1) be in the class $T_n(\lambda, \alpha, \gamma)$ ($0 \le \alpha < 1$, $0 \le \lambda < 1$, $0 \le \gamma \le 1/2$ and $n \in N_0$). Then $f_1 * f_2 * f_3(z) \in T_n(\lambda, \xi(n, \lambda, \alpha, \gamma), \gamma)$, where*

$$\xi(n, \lambda, \alpha, \gamma) = \tag{8.21}$$

$$1-\frac{(1-\lambda)(1-\alpha)^3}{4^{n+1}\{2-\alpha(1+\lambda)\}^3(1-\gamma)^2-(1+\lambda)(1-\alpha)^3}.$$

The result is best possible for the functions

$$f_j(z)=z-\frac{(1-\alpha)}{2^{n+1}\{2-\alpha(1+\lambda)\}(1-\gamma)}z^2 \quad (j=1,\ 2,\ 3). \tag{8.22}$$

Proof. From Theorem 11, we have $f_1 * f_2(z) \in T_n(\lambda, \beta(n,\lambda,\alpha,\gamma),\gamma))$, where $\beta(n,\lambda,\alpha,\gamma)$ is given by (8.2). We now use Theorem 12, we get $f_1 * f_2 * f_3(z) \in T_n(\lambda, \xi(n,\lambda,\alpha,\gamma),\gamma)$, where

$$\xi(n,\lambda,\alpha,\gamma)=$$
$$1-\frac{(1-\lambda)(1-\alpha)(1-\beta)}{2^n c(\gamma,2)\{2-\alpha(1+\lambda)\}\{2-\beta(1+\lambda)\}-(1+\lambda)(1-\alpha)(1-\beta)},$$

$$\xi(n,\lambda,\alpha,\gamma)=1-\frac{(1-\lambda)(1-\alpha)^3}{4^{n+1}\{2-\alpha(1+\lambda)\}^3-(1+\lambda)(1-\alpha)^3}.$$

This completes the proof of Corollary 8.

THEOREM 13. *Let the functions $f_j(z)$ $(j=1,\ 2)$ defined by* (5.1) *be in the class $T_n(\lambda,\alpha,\gamma)$ ($0\le\alpha<1$, $0\le\lambda<1$, $0\le\gamma\le 1/2$ and $n\in N_0$). Then the function*

$$h(z)=z-\sum_{k=2}^{\infty}\left(a_{k,1}^2+a_{k,2}^2\right)\ z^k \tag{8.23}$$

belongs to the class $T_n(\lambda,\psi(n,\lambda,\alpha,\gamma),\gamma)$ where

$$\psi(n,\lambda,\alpha,\gamma)=1-\frac{(1-\lambda)(1-\alpha)^2}{2^n\{2-\alpha(1+\lambda)\}^2(1-\gamma)-(1+\lambda)(1-\alpha)^2}. \tag{8.24}$$

The result is sharp for the functions $f_j(z)$ $(j=1,\ 2)$ defined by (8.14).

Proof. By virtue of Theorem 1, we obtain

$$\sum_{k=2}^{\infty}\left[\frac{k^n\{k-\alpha[1+\lambda(k-1)]\}}{(1-\alpha)}\right]^2 [c(\gamma,k)]^2\, a_{k,1}^2 \le \left[\sum_{k=2}^{\infty}\frac{k^n\{k-\alpha[1+\lambda(k-1)]\}}{(1-\alpha)}\,c(\gamma,k)\,a_{k,1}\right]^2 \le 1 \tag{8.25}$$

and

$$\sum_{k=2}^{\infty}\left[\frac{k^n\{k-\alpha[1+\lambda(k-1)]\}}{(1-\alpha)}\right]^2 [c(\gamma,k)]^2\, a_{k,2}^2 \le \left[\sum_{k=2}^{\infty}\frac{k^n\{k-\alpha[1+\lambda(k-1)]\}}{(1-\alpha)}\,c(\gamma,k)\,a_{k,2}\right]^2 \le 1. \tag{8.26}$$

It follows from (8.25) and (8.26) that

$$\sum_{k=2}^{\infty} \frac{1}{2} \left[\frac{k^n\{k-\alpha[1+\lambda(k-1)]\}}{(1-\alpha)}\right]^2 [c(\gamma,k)]^2 \left(a_{k,1}^2 + a_{k,2}^2\right) \leq 1. \tag{8.27}$$

Therefore we need to find the largest $\psi = \psi(n, \lambda, \alpha, \gamma)$ such that

$$\frac{k^n\{k-\psi[1+\lambda(k-1)]\}}{(1-\psi)} \leq \frac{1}{2} \left[\frac{k^n\{k-\alpha[1+\lambda(k-1)]\}}{(1-\alpha)}\right]^2 c(\gamma,k) \quad (k \geq 2). \tag{8.28}$$

that is, that

$$\psi \leq 1 - \frac{2(1-\lambda)(k-1)(1-\alpha)^2}{k^n\{k-\alpha[1+\lambda(k-1)]\}^2 c(\gamma,k) - 2[1+\lambda(k-1)](1-\alpha)^2} \quad (k \geq 2). \tag{8.29}$$

Since

$$D(k) = 1 - \frac{2(1-\lambda)(k-1)(1-\alpha)^2}{k^n\{k-\alpha[1+\lambda(k-1)]\}^2 c(\gamma,k) - 2[1+\lambda(k-1)](1-\alpha)^2} \quad (k \geq 2) \tag{8.30}$$

is an increasing function of k $(k \geq 2)$ for $0 \leq \alpha < 1$, $0 \leq \lambda < 1$, $0 \leq \gamma \leq 1/2$ and $n \in N_0$, we readily have

$$\psi \leq 1 - \frac{(1-\lambda)(1-\alpha)^2}{2^n\{2-\alpha[1+\lambda)]\}^2(1-\gamma) - (1+\lambda)(1-\alpha)^2} \tag{8.31}$$

which completes the proof of Theorem 13.

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