# Dual Zassenhaus Lemma

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**Abstract**

Zassenhaus Lemma can be easily proved in modular lattices. Consequently, the dual result also holds even though it is nowhere mentioned ! However the group theoretic version needs some comments.

## 1 Introduction

Recall that in an arbitrary lattice $L$ two intervals $A$, $B$ are called *similar* (or *transposed*) if there are elements $a, b \in L$ such that $\{A, B\} = \{[a \wedge b, a], [b, a \vee b]\}$ and *projective* if there are intervals $A = I_0, I_1, .., I_n = B$ such that $I_{k-1}$ and $I_k$ are similar for every $1 \leq k \leq n$.

Two chains

$$a = a_0 \leq a_1 \leq ... \leq a_m = b \ (1)$$
$$a = b_0 \leq b_1 \leq ... \leq b_n = b \ (2)$$

between the same two elements $a$, $b$ of $L$ are called *equivalent* if $m = n$ and there is a permutation $\sigma \in S_n$ such that the intervals $[a_{i-1}, a_i]$ and $[b_{\sigma(i)-1}, b_{\sigma(i)}]$ are projective.

*In a modular lattice, every two similar and hence any two projective intervals are isomorphic.*

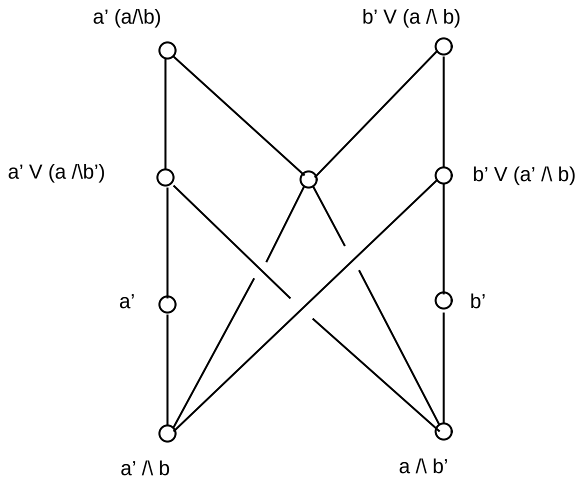
Indeed, the maps $- \vee b : [a \wedge b, a] \to [b, a \vee b]$ and $a \wedge - : [b, a \vee b] \to [a \wedge b, a]$ are lattice isomorphisms (inverse to each other).

First, let us state and prove the

**Theorem 1.1** (Zassenhaus) *Let $a' \leq a$, $b' \leq b$ be elements in a modular lattice. The intervals $[a' \vee (a \wedge b'), a' \vee (a \wedge b)]$ and $[b' \vee (a' \wedge b), b' \vee (a \wedge b)]$ are projective (and hence isomorphic).*

*Proof.* We shall show that the intervals $[a' \vee (a \wedge b'), a' \vee (a \wedge b)]$ and $[(a' \wedge b) \vee (a \wedge b'), a \wedge b]$ are similar. Symmetrically, $[b' \vee (a' \wedge b), b' \vee (a \wedge b)]$ and $[(a' \wedge b) \vee (a \wedge b'), a \wedge b]$ are similar and so the claim follows.

Actually (for groups, see [1]) the following diagram describes our situation (the elements in brackets refer to the proof of the next Theorem)




The equality $(a' \vee (a \wedge b')) \vee (a \wedge b) = a' \vee (a \wedge b)$ is straightforward, and $(a' \vee (\underline{a \wedge b'})) \wedge (\underline{a \wedge b}) = (a \wedge b') \vee (a' \wedge (a \wedge b)) = (a \wedge b') \vee (a' \wedge b)$ follows using modularity. $\square$

**Remark.** For modules the isomorphism part can be (see [3]) also proved (and hence in exact categories) using (two times) the 9-Lemma.

A *refinement* of a chain is a chain obtained from this by inserting new elements.

We obtain immediately the lattice version of :

**Theorem 1.2** (Schreier) *In a modular lattice, any two chains between the same two elements have equivalent refinements.*

*Proof.* For two chains $a = a_0 \le a_1 \le .. \le a_m = b$ and $a = b_0 \le b_1 \le .. \le b_n = b$ we denote by $a_{ij} = (a_i \wedge b_j) \vee b_{j-1}$ respectively $b_{ji} = (b_j \wedge a_i) \vee a_{i-1}$ for each $i \in \{1, 2, ..., m\}$, $j \in \{1, 2, ..., n\}$. Using (repeatedly) Zassenhaus theorem (just take $a' = a_{i-1}$, $a = a_i$ respectively $b' = b_{j-1}$ , $b = b_j$), $[a_{i-1,j}, a_{ij}]$ and $[b_{j-1,i}, b_{ji}]$ are projective. Hence, from the chains (1) and (2) we obtain the chains

$$\begin{array}{c} a = a_{01} \le a_{11} \le a_{21} \le ... \le a_{m1} \le a_{12} \le ... \le a_{mn} = b \ (3) \\ a = b_{01} \le b_{11} \le b_{21} \le ... \le b_{n1} \le b_{12} \le ... \le b_{nm} = b \ (4) \end{array} .$$

But $a_i = b_{ni}$ and $b_j = a_{mj}$ so that (4) is a refinement of (1) and (3) is a refinement of (2). Moreover, (3) and (4) are equivalent. $\square$

**Definition.-** A chain between two elements $a$ and $b$ is called a *composition chain* if

$$a = a_0 < a_1 < .. < a_n = b$$

has no refinements (i.e. $[a_{k-1}, a_k] = \{a_{k-1}, a_k\}$ for every $1 \le k \le n$).

The last theorem now gives

**Theorem 1.3** (Jordan-Hölder) *In a modular lattice, any two composition chains between the same two elements are equivalent.* $\square$

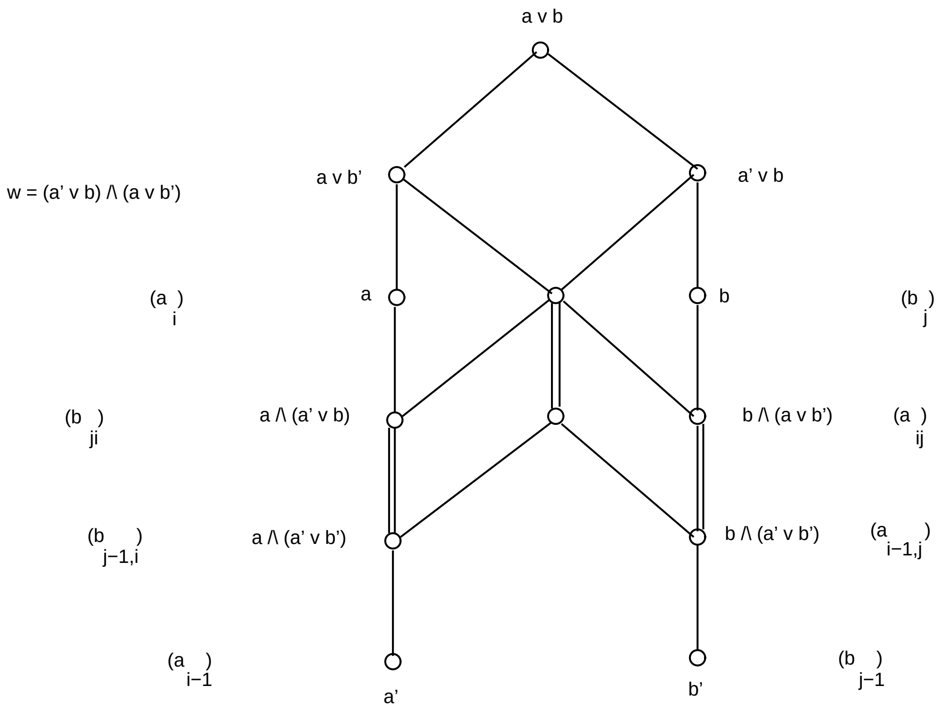
Therefore in the above definition, $n$ can be called the *length of the interval* $[a, b]$ denoted by $l[a, b]$. As a special case, in a modular lattice $L$ with 0 and 1 we use the *length* $l(L)$ *of the lattice* $L$ as $l[0, 1]$ if a finite chain between 0 and 1 exists (in this case we say that $L$ has *finite length*).

Clearly (**but nowhere mentioned in the literature**) one dually proves the

**Theorem 1.4** (Zassenhaus dual) *Let $a' \leq a$, $b' \leq b$ be elements in a modular lattice. The intervals $[a \wedge (a' \vee b'), a \wedge (a' \vee b)]$ and $[b \wedge (a' \vee b'), b \wedge (a \vee b')]$ are projective (and hence isomorphic).* $\square$

Moreover, this dual makes possible a dual proof of Schreier theorem:

For the chains (1) and (2) we select the elements $a_{ij} = b_j \wedge (a_i \vee b_{j-1})$ and $b_{ji} = a_i \wedge (a_{i-1} \vee b_j)$ . Now the refinements are described by the diagram




# 2 Back to groups

In the last generation books on group theory the reader can find more or less exactly the above procedure in order to define the (sometimes called) *composition length* of a group $G$ (as consequence of the corresponding Jordan-Hölder theorem). Of course, this procedure is also used for $R$-modules over a ring with identity $R$, in order to define the *length of the module*.

The older books define the isomorphism in Zassenhaus Lemma and heavily check that it is independent on the representatives selection...

The group version of the result we discuss is the well-known

**Zassenhaus Lemma**: *Let $H'$, $H$ , $K'$, $K$ be subgroups of a group $G$ such that $H'$ is normal in $H$ and $K'$ is normal in $K$. Then $H'(H \cap K')$ is a normal subgroup of $H'(H \cap K)$, $K'(H' \cap K)$ is a normal subgroup of $K'(H \cap K)$ and the corresponding factor groups are isomorphic, i.e.*

$$\frac{H'(H \cap K)}{H'(H \cap K')} \simeq \frac{K'(H \cap K)}{K'(H' \cap K)}.$$

*Some comments about the corresponding proof.*

The isomorphism between the two factor groups is obtained using the above procedure if someone 'covers' in some way the normality of the subgroups involved. The normality is indeed essential: only the sublattice of all the normal subgroups of an arbitrary group is modular!

What about the isomorphisms pointed out above?

$a \wedge - : [b, a \vee b] \to [a \wedge b, a]$ is now covered by (i) $H \leq G$ and $K' \lhd K$ imply $H \cap K' \lhd H \cap K$.

As for the isomorphism $- \vee b : [a \wedge b, a] \to [b, a \vee b]$ there are some troubles because $S \lhd T$ implies $NS \lhd NT$ only if $N$ is also normal in $G$ (take for instance two subgroups of order 2 resp. 3 in $A_4$).

But luckily: $H' \lhd H$ and $K' \lhd K$ imply $H \cap K' \lhd H \cap K$ which are both subgroups in $H$ where $H'$ **is** normal! So we can continue by $H'(H \cap K') \lhd H'(H \cap K)$ (and symmetrically $K'(H' \cap K) \lhd K'(H \cap K)$).

As for the intermediate factor group $\dfrac{H \cap K}{(H \cap K')(H' \cap K)}$ both $H \cap K'$ and $H' \cap K$ are normal in $H \cap K$ so that $(H \cap K')(H' \cap K)$ is (a subgroup and) normal in $H \cap K$.

Once again, in the group theory literature known to the author there is no Dual Zassenhaus Lemma for groups. To a superficial analysis the reason is clear: for subgroups $H'$, $H$ , $K'$, $K$ of a group $G$ such that $H'$ is normal in $H$ and $K'$ is normal in $K$, $H'K'$, $H'K$ and $HK'$ may not even be subgroups of $G$, so no question of $H \cap (H'K')$ is a normal subgroup of $H \cap (H'K)$, $K \cap (H'K')$ is a normal subgroup of $K \cap (HK')$ and the corresponding isomorphism.

However, with a slight modification, this dual can be safeguarded!

**Zassenhaus Dual Lemma**: *Let $H' \leq H$ , $K' \leq K$ be subgroups of a group $G$ such that $H'$ and $K'$ are normal in $G$. Then $H \cap (H'K')$ is a normal subgroup of $H \cap (H'K)$, $K \cap (H'K')$ is a normal subgroup of $K \cap (HK')$ and the corresponding factor groups are isomorphic, i.e.*

$$\frac{H \cap (H'K)}{H \cap (H'K')} \simeq \frac{K \cap (HK')}{K \cap (H'K')}.$$

Once again only the normality part needs comments:

if $H' \lhd G$ and $K' \lhd G$ then also $H' \lhd H$ and $K' \lhd K$, $H'K'$ is a subgroup, normal in $H'K$ respectively $HK'$ and we obtain $H \cap (H'K') \lhd H \cap (H'K)$, symmetrically $K \cap (H'K') \lhd K \cap (HK')$ respectively $H'K' \lhd (H'K)(HK')$ for the intermediate factor group.

# 3 Unfortunately, no Applications

Is the Dual Zassenhaus Lemma useful ?

1) Composition series are usually defined taking subnormal series. Consequently, this result **cannot be used** in proving Schreier and hence Jordan-Hölder Theorems for groups.

2) In [2], subgroups $U$ of direct products $G = H \times K$ of two groups are characterized using sections, isomorphisms and diagonals. Of special importance is the isomorphism

$$\frac{UK \cap H}{U \cap H} \simeq \frac{UH \cap K}{U \cap K}.$$

Here again the isomorphism is defined and one heavily checks that it is independent on the representatives selection.

Notice that **part of this** can be easily derived from the Zassenhaus Dual Lemma, taking $U = H'K'$ (i.e., for the decomposable subgroups).

3) For modules and abelian groups, this gives for arbitrary submodules $H' \leq H$ and $K' \leq K$ of an $R$-module $M$, the isomorphism

$$\frac{H \cap (H' + K)}{H \cap (H' + K')} \simeq \frac{K \cap (H + K')}{K \cap (H' + K')}.$$

Using this one derives at once (take $K = M$ and $H' \leq K'$) the second Nother isomorphism Theorem.

# References


[1] Robinson D.J.S., *A Course in the Theory of Groups*, Second Edition, Graduate Texts in Mathematics, 80, Springer - Verlag, New York Inc., 1995.

[2] Schmidt R., *Subgroup lattices of groups*. de Gruyter Expositions in Mathematics, 14. Walter de Gruyter.

[3] Wisbauer R., *Foundations of Module and Ring Theory*, Gordon and Breach, 1991.