ABELIAN GROUPS WITH UNIT-REGULAR ENDOMORPHISM RING

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Abstract. Old and recent results permit to give comprehensive information on Abelian groups with unit-regular endomorphism ring. Mostly all such Abelian groups are determined.

A ring R is (Von Neumann) *regular* if for every $a \in R$ there is an element $b \in R$ such that $a = aba$, and *unit regular* (see Ehrlich [2]), if b is a unit. It is *strongly regular* if for every $a \in R$ there is an element $b \in R$ with $a = a^2b$. It is Dedekind finite if $ba = 1$ whenever $ab = 1$ for $a, b \in R$.

1. After 1976,

when Ehrlich's paper appeared ([3]), the following information about Abelian groups with unit-regular endomorphism rings (hereafter called *unit-regular groups*) was available:

1. *Let V be a vector space over a division ring D and $R = \mathrm{End}_D V$. R is unit-regular if and only if V is finite dimensional.*

2. *Let A be a ring with identity, M a right A-module such that $R = \mathrm{End}_A M$ is regular and suppose M is a direct sum of isomorphic indecomposable submodules of M. Then R is unit-regular if and only if M is a direct sum of finitely many isomorphic indecomposable submodules.*

3. *Let A be a ring with identity, M a right A-module such that $M = \bigoplus_{i \in I} M_i$, where each M_i is a fully invariant submodule, equal to a direct sum of isomorphic indecomposable submodules. $R = \mathrm{End}_A M$ is unit-regular if and only if it is Dedekind finite.*

Since unit-regular rings are regular, the following Fuchs, Rangaswamy results (see Proposition 112.7 [4]) were also useful

(a) If G is not reduced, then $\mathrm{End}(G)$ is regular if and only if G is a direct sum of a torsion-free divisible group and an elementary group.
(b) If G is torsion, $\mathrm{End}(G)$ is regular if and only if G is elementary.
(c) If G is reduced and $\mathrm{End}(G)$ regular, then $T(G)$ (the torsion part of G) is elementary, $G/T(G)$ is torsion-free divisible and $\bigoplus G_p \leq G \leq \prod G_p$.

Combining these results gives at once the following

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Theorem A. *Let G be an Abelian group.*

1. If G is torsion, $\mathrm{End}(G)$ *is unit-regular if and only if G is elementary, with (any number of) finite p-components;*

2. If G is not reduced, $\mathrm{End}(G)$ *is unit-regular if and only if G is a direct sum of an elementary group with finite p-components [as in 1], and a torsion-free divisible group of finite rank;*

3. If G is reduced and $\mathrm{End}(G)$ *is (unit-)regular then $T(G)$ is elementary, $G/T(G)$ is torsion-free divisible and $\bigoplus G_p \leq G \leq \prod G_p$.*

Rephrasing some of these results yields the following

Summary 1. *(i) The following conditions are equivalent: (a) G is torsion-free unit-regular; (b) G is divisible unit-regular; (c) $G \cong \mathbf{Q} \oplus \mathbf{Q} \oplus ... \oplus \mathbf{Q}$, i.e., a finite direct sum of $\mathbf{Q}$.*

(ii) If G is torsion, G is unit-regular if and only if G is elementary, with (any number of) finite p-components.

(iii) If G is not reduced, then G is unit-regular if and only if G is (splitting and) a direct sum of a finite rank torsion-free divisible group and an elementary group with finite primary components.

Examples. $\mathbf{Q}$ and $\mathbf{Z}(p)^n$ are unit-regular and, for $n > 1$, $\mathbf{Z}(p^n)$ is not unit-regular.

Both $\mathbf{Q}$ and $\prod\limits_p \mathbf{Z}(p)$ are unit-regular, but $M = \mathbf{Q} \oplus \prod\limits_p \mathbf{Z}(p)$ is not unit-regular (it is known that $\mathrm{End}(M)$ is 2-regular but not regular).

$U = \prod\limits_p \mathbf{Z}(p)$ is unit-regular since, as direct product $\prod\limits_p \mathbf{Z}_p$ of fields, the endomorphism ring of U is unit-regular (indeed, every field is unit-regular, and a direct product of rings is unit-regular if and only if each factor is unit-regular).

Further, for $\mathrm{End}(G)$ unit-regular, it was no real hope to improve these results in the reduced case. Indeed

$$\mathrm{strongly-regular} \Longrightarrow \mathrm{unit-regular} \Longrightarrow \mathrm{regular}$$

and this undecided situation (a reduced group having regular endomorphism rings lies between the direct sum and the direct product of its p-components, which are elementary groups, but singling out these groups was not possible) remains also in the strongly regular case (see Lemma 112.10, Proposition 112.8 [[4]]):

For a group $G = C \oplus D$ with C reduced and D divisible, $\mathrm{End}(G)$ *is strongly regular if and only if* $\mathrm{End}(C)$ *is strongly regular and D is torsion-free of finite rank.*

If $\mathrm{End}(C)$ *is strongly regular, it has finite elementary p-components, $C/T(C)$ is (torsion-free) divisible of finite rank and $\bigoplus C_p \leq C \leq \prod C_p$.*

2. After morphic

An endomorphism α of a module ${}_RM$ is called *morphic* if $M/\mathrm{im}\alpha \cong \ker\alpha$, that is, if the dual of the Noether isomorphism theorem holds for α. The module ${}_RM$ is called *morphic* if every endomorphism is morphic. In 1976, Ehrlich [3] showed that an endomorphism α of a module ${}_RM$ is unit-regular if and only if it is regular and morphic.

In passing recall, *corner rings of unit-regular rings are unit-regular* (see [8], **Ex.21.9**). Hence, the class of modules having unit-regular endomorphism rings is closed under direct summands. More, *unit-regularity is a Morita invariant*: it passes to matrix rings and to full corner rings.

After defining morphic modules, Nicholson and Campos ([10]) showed that

$$\mathrm{End}_R(M) \text{ unit-regular } \Longrightarrow M \text{ morphic.}$$

Recently, the second author of this note has determined (see [1]) mostly all the morphic Abelian groups (i.e., $\mathbf{Z}$-modules), and these are somewhat rare, so that the above implication gave good prospects in determining the Abelian groups with unit-regular endomorphism ring.

In order to give the reader a better perspective, before starting we mention the following chart

$$\begin{array}{lcll} & \nearrow & G \text{ morphic} & \longrightarrow \ \mathrm{End}(G) \text{ Dedekind finite} \\ \mathrm{End}(G) \text{ unit-regular} & & & \\ & \searrow & \mathrm{End}(G) \text{ regular} & \end{array}$$

and $\mathrm{End}(G)$ strongly regular $\longrightarrow$ $\mathrm{End}(G)$ unit-regular.

Using the new results from [1] we can easily dispose of the splitting mixed case. Indeed, recall

- [4]: *if* $\mathrm{End}(G)$ *is strongly regular, then* $G = C \oplus D(G)$ *with the divisible part* $D(G)$ *torsion-free of finite rank and* C *a reduced group with finite elementary* p*-components,* $C/T(C)$ *is (torsion-free) divisible and* $\bigoplus C_p \leq C \leq \prod C_p$.
- [1]: *the splitting morphic mixed groups are exactly the groups* $G = T(G) \oplus D(G) = \bigoplus_p (\mathbf{Z}(p^{k_p})^{n_p} \oplus \mathbf{Q}^k$ *with nonnegative integers* k_p, n_p *and* k.

Therefore

Theorem 2. *The splitting unit-regular mixed groups are exactly the groups* $G = T(G) \oplus D(G) = \bigoplus_p (\mathbf{Z}(p))^{n_p} \oplus \mathbf{Q}^k$ *with nonnegative integers* n_p *and* k.

Moreover, again using [1], we can prove

Theorem 3. *If* G *is a reduced unit-regular group then the primary components* G_p *are (elementary) and finite.*

Proof. Notice that by our previous considerations, only the claim outside the brackets has to be proved.

First, if G is unit-regular then G is morphic. By [1], $T(G)$ is also morphic and so has finite homogeneous primary components. In our case these must be elementary and finite.

Next, we have to show that the rank of (the torsion-free divisible group) $G/T(G)$ is finite. □

Therefore

Summary 4. *If G is unit-regular, then $G = C \oplus D(G)$ with torsion-free of finite rank divisible part $D(G)$ and C reduced group with finite elementary p-components, $C/T(C)$ is (torsion-free) divisible and $\bigoplus C_p \leq C \leq \prod C_p$.*

Conjecture 5. *As in the strongly regular case, if G is a reduced unit-regular group then $G/T(G)$ has finite rank.*

Examples can be given in order to show that generally $G/T(G)$ fails to be unit-regular whenever G is unit-regular.

Example 6. *The group $G = \prod_p \mathbf{Z}(p)$ is unit-regular, but $G/T(G)$ is not unit-regular (nor splitting).*

As direct product $\prod_p \mathbf{Z}_p$ of fields, the endomorphism ring of G is unit-regular (again, every field is unit-regular, and a direct product of rings is unit-regular if and only if each factor is unit-regular), but $G/T(G)$ is not DF (nor unit-regular: it is infinite rank torsion-free divisible).

More, we can answer in the negative a natural question: is $G = T(G) \oplus F$ unit-regular if both $T(G)$ and $F = G/T(G)$ are both unit-regular?

Example 7. *Let $H = \mathbf{Q} \oplus P$ with the subgroup $P = P(U, a) = \{g \in U : ng \in \langle a \rangle$ for some positive integer $n\}$ of all elements in U that depend on $\{a\}$ (here once again $U = \prod_p \mathbf{Z}(p)$ and a is the infinite order element $(\overline{1}, \overline{1}, ...)$. Then $T(H)$ and $H/T(H)$ are both unit-regular, but H is not unit-regular.*

3. Inside Γ

Singling out the (reduced) groups G which share the property $\bigoplus G_p \leq G \leq \prod G_p$ is a long-standing unsolved problem in Abelian group Theory. Thus, for the time being, there is no hope to give a complete characterization of the (mixed reduced) unit-regular groups (recall that this undecidable situation lasts also in the strongly regular case).

However it is worth mentioning a celebrated environment, a class of groups which was under close scrutiny the last 15 years, for Abelian group theorists. In [9], a class of reduced mixed groups of finite torsion-free rank, denoted Γ was defined for the study of regular or PP (principal projective) endomorphism rings of mixed (Abelian) groups, as follows: $G \in \Gamma$ if there is a pure embedding $\bigoplus G_p < G < \prod G_p$.

Then it can be proved

Lemma 8. *A reduced (mixed) group G of finite torsion-free rank belongs to Γ if and only if for all primes p, the p-component is a direct summand of G, and, $G/T(G)$ is divisible.*

Therefore, using our previous results (every p-component of a unit-regular group is pure and bounded, so a direct summand) we obtain at once

Proposition 9. *Every unit-regular reduced (mixed) group G of finite torsion-free rank belongs to Γ.*

Therefore, the most we can do is to single out such groups inside Γ.
First

Theorem 10. *If G a finite torsion-free rank morphic (mixed reduced) group then $G = T + A$ with torsion T such that $T_p \neq 0$ iff $A_p = 0$ and, $A \in \mathcal{G}$, that is, A is self-small in Γ.*

Proof. ... □

Theorem 11. *If G a finite torsion-free rank unit-regular (mixed reduced) group then $G = T + A$ with torsion T such that $T_p \neq 0$ iff $A_p = 0$, $A \in \mathcal{G}$, that is, A is self-small in Γ and ?...*

Proof. ... □

Theorem 12. *If G a finite torsion-free rank strongly regular (mixed reduced) group then $G = T + A$ with torsion T such that $T_p \neq 0$ iff $A_p = 0$, $A \in \mathcal{G}$, that is, A is self-small in Γ and ??...*

Proof. ... □

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