Realization theorems for triangular rings

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Abstract

Triangular matrix rings which are endomorphism rings of Abelian groups are characterized and Abelian groups with (or without) triangular endomorphism rings are investigated.

While every ring is the endomorphism ring of some module, "in investigations on endomorphism rings, a general trend is to find conditions on an abstract ring to be the endomorphism ring of some Abelian group. It seems to be a rather difficult problem to obtain necessary and sufficient conditions in general, but in a few special cases fairly satisfactory answer is known" ([1]). In this paper, a solution is given for a special class of rings: the (formal) triangular rings.

We can ask two questions.

First, when is a given triangular ring isomorphic to the endomorphism ring of an Abelian group?

Secondly, when is the endomorphism ring of a given Abelian group isomorphic to a triangular ring?

Just to simplify the wording, groups will be called *triangular* (or not triangular) if they have (no) triangular endomorphism rings. It turns out that there are plenty of classes of triangular groups as well as plenty of classes of not triangular groups.

A simple Theorem gives a complete answer to the first question. As for the second, complete characterizations are given up to reduced torsion-free and reduced nonsplitting mixed Abelian groups. For Abelian groups our results can be summarized as follows:

1) A divisible group is not triangular if and only if it is a p-group or it is torsion-free.

2) A reduced torsion group is not triangular if and only if it is a p-group, for some prime number p.

3) All genuine splitting mixed groups are triangular.

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4) All not divisible nor reduced groups (i.e., $G = D(G) \oplus R$ with divisible part $D(G)$ and $D(G) \neq 0 \neq R$) are triangular.

5) Among finite rank torsion-free groups, the irreducible groups (in J. Reid's sense) are not triangular.

All rings considered are (associative) with identity, all bimodules are unitary and all groups are Abelian.

1 The realization

The answer to the first question is given by the following

Theorem 1 *Let A, B be nonzero rings, ${}_AC_B$ be a bimodule and $S = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ be the (formal) triangular ring. S is isomorphic with the endomorphism ring of a group if and only if A and B are isomorphic with endomorphism rings of groups, say, H and K, such that* $\mathrm{Hom}(H, K) = 0$ *and* $C \cong \mathrm{Hom}(K, H)$, *as bimodules.*

Proof. Since the conditions are known to be sufficient (and in this case $S \cong \mathrm{End}(H \times K)$), assume S is isomorphic to the endomorphism ring of a group G. Then G becomes a left S-module ${}_SG$, and S has two orthogonal idempotents $e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq 0 \neq e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ with $e_{11} + e_{22} = 1$, the identity 2×2 matrix. Consequently, G has a direct decomposition $G = e_{11}G \oplus e_{22}G = H \oplus K$. Since for two idempotents ε, ω in a ring $\mathrm{End}(G)$, there exist a canonical group isomorphism $\mathrm{Hom}(\omega G, \varepsilon G) \cong \varepsilon \mathrm{End}(G)\omega$, we deduce $\mathrm{Hom}(H, K) = \mathrm{Hom}(e_{11}G, e_{22}G) \cong e_{11}\mathrm{End}(G)e_{22} = 0$, and $\mathrm{Hom}(K, H) = \mathrm{Hom}(e_{22}G, e_{11}G) \cong C$, by simple computation and the proof is complete. ■

Remarks. 1) Since $H = e_{11}G \neq 0 \neq e_{22}G = K$, the group G is decomposable.

2) The hypothesis "nonzero" rings is added in order to avoid trivial realizations as $A \cong \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ or $B \cong \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}$.

By denial we obtain some useful tools which generate simple examples:

(i) *Suppose A and B are endomorphism rings of some groups, and for every groups H and K with $A \cong \mathrm{End}(H)$, $B \cong \mathrm{End}(K)$, $\mathrm{Hom}(H, K) \neq 0$. Then the triangular matrix ring $R = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ is not isomorphic to the endomorphism ring of any group.*

Example: $\begin{bmatrix} \mathbf{Z} & \mathbf{Q} \\ 0 & \mathbf{Q} \end{bmatrix}$ is not isomorphic to the endomorphism ring of any group.

Indeed, $\mathrm{End}(\mathbf{Q}) \cong \mathbf{Q}$ and it is the only one (see [3]), $\mathrm{End}(\mathbf{Z}) \cong \mathbf{Z}$ is not the only one but $\mathbf{Z} \cong \mathrm{End}(H)$ implies, H torsion-free, reduced and rigid. Hence $\mathrm{Hom}(H, \mathbf{Q}) \cong \prod_{r_0(H)} \mathbf{Q}$, so $\neq 0$. Clearly, $\mathrm{Hom}(\mathbf{Z}, \mathbf{Q}) \cong \mathbf{Q}$ and $\mathrm{Hom}(\mathbf{Q}, \mathbf{Z}) = 0$.

(ii) *Suppose A and B are endomorphism rings of some groups, and for every groups H and K with $A \cong \mathrm{End}(H)$, $B \cong \mathrm{End}(K)$ and $\mathrm{Hom}(H, \dot{K}) = 0$, the bimodule ${}_AC_B \ncong \mathrm{Hom}(K, H)$. Then the triangular matrix ring $R = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ is not isomorphic to the endomorphism ring of any group.*

Example: $\begin{bmatrix} \mathbf{Q} & \mathbf{Z} \\ 0 & \mathbf{Z} \end{bmatrix}$ is not isomorphic to the endomorphism ring of any group.

2 Not triangular groups

As it is readily seen, it is somehow easier to isolate the groups which **do not** have triangular endomorphism rings. According to the previous Theorem these are exactly the groups G such that for every decomposition $G = H \oplus K$ neither $\mathrm{Hom}(H, K)$ nor $\mathrm{Hom}(K, H)$ vanishes. Since for a direct decomposition $G = H \oplus K$, H is fully invariant in G if and only if $\mathrm{Hom}(H, K) = 0$, a useful characterization follows

Proposition 2 *The not triangular groups are exactly the groups with no proper fully invariant direct summands.*

Little attention has been paid so far to groups without proper fully invariant direct summands (we can eliminate the indecomposable groups by a previous Remark). The general perception among Abelian group theorists is that "fully invariant direct summands are rare and so investigation about fully invariant summands (or lack of it) does not lead to interesting classes of groups". Thus, no study was ever done in this direction. However some (traditional) reductions can be made.

Since $\mathrm{Hom}(A, B) = 0$ for: (i) A is torsion and B is torsion-free, (ii) A is divisible and B is reduced, (iii) A is a p-group and B is a q-group with p, q different prime numbers, it follows at once

Lemma 3 *1) Genuine splitting mixed groups are triangular.*

2) Not triangular groups must be divisible or reduced.

3) Torsion groups with at least two primary components corresponding to different primes are triangular.

and so,

Proposition 4 *(a) A divisible group is not triangular if and only if it is a p-group or it is torsion-free (i.e., $\mathbf{Q} \oplus \mathbf{Q} \oplus \ldots$ or $\mathbf{Z}(p^\infty) \oplus \mathbf{Z}(p^\infty) \oplus \ldots$).*

(b) A reduced torsion group is not triangular if and only if it is a p-group.

Proof. (b) Indeed, the only fully invariant pure subgroups in a p-group G are $0, G$ and the divisible part $D(G)$. Thus reduced p-groups are not triangular. ■

Therefore, what is left for the determination of groups with no proper fully invariant direct summands (or not triangular groups) are the *reduced torsion-free groups*, respectively the *non-splitting reduced mixed groups*.

3 The reduced torsion-free case

A subclass of all the reduced torsion-free groups with no fully invariant (proper) direct summands was studied by Reid [2] in the early 60's. Torsion-free groups with no proper fully invariant pure subgroups were called *irreducible*.

For a torsion-free group G, $\mathbf{Q}\otimes\mathrm{End}(G)$ is called the *quasi-endomorphism ring* of G (we shall simply write $\mathrm{QEnd}(G)$). This is a $\mathbf{Q}$-vector space whose dimension is finite whenever the group G has finite rank. Two torsion-free groups of finite rank are called *quasi-isomorphic* if each is isomorphic to a subgroup of finite index of the other group. The group G is *strongly indecomposable* if A is not quasi-isomorphic to the direct sum of two non-zero groups.

Theorem 5 *(Reid [2]) For a torsion-free group G of finite rank, the following are equivalent:*

(i) G is irreducible;

(ii) G is quasi-isomorphic to a finite direct sum of isomorphic, strongly indecomposable irreducible groups;

(iii) the quasi-endomorphism ring $\mathrm{QEnd}(G)$ *of G is the full $m \times m$ matrix ring (for some integer m) over a division ring D; and the rank of G is md where d denotes the $\mathbf{Q}$-dimension of D.*

Moreover, *an irreducible group of finite rank is strongly indecomposable exactly if its quasi-endomorphism ring is a division ring.*

So finally, we just mention that all the *irreducible finite rank torsion-free groups are not triangular.*

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References

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