A characterization of Dedekind finite rings

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Abstract

A unital ring R is called *Dedekind finite* if $ba = 1$ whenever $ab = 1$, for $a, b \in R$.

A ring R is called *stably* (or weakly) *finite* if $\mathcal{M}_n(R)$ is Dedekind finite for every natural number n. Since $n = 1$ is allowed in this definition, clearly *stably finite implies Dedekind finite.*

A *natural generalization* would be obtained by replacing the (full) matrix ring $\mathcal{M}_n(R)$ by its subring $\mathcal{T}_n(R)$, which consists in all the upper triangular n by n matrices with entries in R.

In this short note we show that this generalization is too large: this class of rings coincides with the class of all Dedekind rings.

Actually we can prove

Theorem 1 *A ring R is Dedekind finite iff $\mathcal{T}_2(R)$ is Dedekind finite iff $\mathcal{T}_n(R)$ are Dedekind finite for every natural number n.*

Proof. Suppose $AX = I_2$ with $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $X = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ for a DF ring R. Then $ax = cz = 1$, $ay + bz = 0$ and by DF, $xa = zc = 1$ so these are (two-sided) units. By left multiplication with x and right multiplication by c, from $ay + bz = 0$ we get $xb + yc = 0$. Hence indeed $XA = I_2$. Conversely, for every six elements a, b, c, x, y, z, the conditions $ax = cz = 1$, $ay + bz = 0$ imply $xa = zc = 1$ and $xb + yc = 0$.

For given $ax = 1$, take $c = z = 1$ and $b = -ay$. Then $xa = 1$, so R is DF.

As for the second equivalence, an induction applies: we suppose that for a Dedekind finite ring R, $\mathcal{T}_{n-1}(R)$ is Dedekind finite and we show that $\mathcal{T}_n(R)$ is also Dedekind finite. We use block decompositions for n by n matrices, that is, in $\mathcal{T}_n(R)$ we take $A_n = \begin{bmatrix} A_{n-1} & \alpha \\ 0 & a_n \end{bmatrix}$ with triangular $A_{n-1} \in \mathcal{T}_{n-1}(R)$, $n-1$-column $\alpha = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$ and $a_n \in R$ and similarly B_n with blocks B_{n-1}, β and b_n.

Then $A_nB_n = I_n$ amounts to $A_{n-1}B_{n-1} = I_{n-1}$, $a_nb_n = 1$ and

(1) $A_{n-1}\beta + \alpha b_n = 0$.

Since R is DF and by induction hypothesis, $B_{n-1}A_{n-1} = I_{n-1}$, $b_na_n = 1$ and in order to get $B_nA_n = I_n$ we only need to show that

(2) $B_{n-1}\alpha + \beta a_n = 0$.

The equalities are indeed equivalent: from (1) we get (2) by left multiplication with B_{n-1}, and right multiplication with a_n. ■