

## STRONGLY INVARIANT SUBGROUPS

GRIGORE CĂLUGĂREANU

Babeş-Bolyai University, Cluj-Napoca, Romania  
E-mail: calu@math.ubbcluj.ro

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**Abstract.** As a special case of fully invariant subgroups, strongly invariant subgroups are introduced and studied for Abelian groups.

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**1. Introduction.** Introduced in 1933 by F. Levi (see [6]) for groups, the role of the fully invariant subobjects in several algebra categories (and especially for groups) is well-known. Suppose  $H \leq K$  are subgroups of a group  $G$  and let  $f$  be an endomorphism of  $G$ . If  $H$  is fully invariant in  $G$  then, by restriction,  $f$  induces an endomorphism of  $H$  and more,  $f$  naturally induces an endomorphism  $\tilde{f}$  of  $K/H$  (defined by  $\tilde{f}(kH) = f(k)H$ ). Now, if we start with a homomorphism  $f : K \rightarrow G$ , in order to induce (also by restriction) an endomorphism of  $H$  and more, to naturally induce (i.e.,  $\tilde{f}(kH) = f(k)H$ ) a homomorphism  $\tilde{f} : K/H \rightarrow G/H$ , a stronger property for  $H$  is needed: this property is defined and studied in the sequel.

The following properties are also well-known:

**FI1)** Fully invariance is transitive.

**FI2)** It is possible to have  $H \leq K \leq G$  with  $H$  fully invariant subgroup of  $G$  but not a fully invariant subgroup of  $K$  [intermediate subgroup property].

**FI3)** The class of all the fully invariant subgroups of a group  $G$  forms a complete sublattice of the subgroup lattice of  $G$ .

**FI4)** The class is commutator closed: if  $H, K$  are fully invariant subgroups of  $G$ , so is the commutator  $[H, K]$ .

**FI5)** If  $H \leq K \leq G$  with  $H$  fully invariant in  $G$  and  $K/H$  fully invariant in  $G/H$ , then  $K$  is fully invariant in  $G$  [quotient transitive].

**FI6)** If  $H$  is fully invariant in  $G$ , then in any finite direct power of  $G^n$ , the corresponding direct power  $H^n$  is fully invariant.

Also well-known and largely used, as a consequence of **FI4**, the derived (or commutator) subgroup  $G' = [G, G]$  and all terms of the lower central series as well as the derived series are fully invariant.

A subgroup  $N$  of a group  $G$  will be called *strongly invariant* in  $G$ , if  $f(N) \leq N$  for every group homomorphism  $f : N \rightarrow G$ . As first examples, the center of the quaternion group  $Q_8$  is a strongly invariant subgroup of  $Q_8$ , but any 4-elements subgroup in  $Q_8$  is not strongly invariant (see diagram on the fourth page of this article). As an Abelian group example consider the rational group  $\mathbf{Q}_p = \{\frac{m}{n} \in \mathbf{Q} \mid \gcd(n, p) = 1\}$ . The multiplication with  $\frac{1}{p}$  shows that  $p\mathbf{Q}_p$  – which is fully invariant in  $\mathbf{Q}_p$  – is not strongly invariant in  $\mathbf{Q}_p$ .

For any endomorphism of  $G$ , just taking the restriction to the subgroup  $N$ , shows that every strongly invariant subgroup is fully invariant. Therefore, we deal only with normal subgroups.

As the previous example shows, the converse fails. The derived subgroup of the dihedral group  $D_{16}$  is another example: it is fully invariant but not strongly invariant.

For any subgroup  $N$  there are some trivial group morphisms which agree with our definition: for the trivial homomorphism  $N \rightarrow G$  obviously  $\{1\} \leq N$  and for the inclusion  $i_N : N \rightarrow G$ , so is  $N \leq N$ . Moreover,  $\{1\}$  and  $G$  are clearly strongly invariant in  $G$ .

Since this is a proper subclass of the class of all fully invariant subgroups of a group, our first concern is what happens with the properties **F11-6** listed above. There are plenty of differences (these statements may be found on the Internet-SubWiki [10] - under an alternative name: *homomorph-containing* subgroup):

**SI1)** Strongly invariance is *not* transitive.

**SI2)** If  $H \leq K \leq G$  with  $H$  a strongly invariant subgroup of  $G$  then  $H$  is also a strongly invariant subgroup of  $K$ .

**SI3)** The class of all the strongly invariant subgroups of a group is closed under arbitrary joins, but *not* under intersections.

**SI4)** The class of all the strongly invariant subgroups of a group is *not* commutator closed.

**SI5)** If  $H \leq K \leq G$  with  $H$  strongly invariant in  $G$  and  $K/H$  strongly invariant in  $G/H$ , then  $K$  is strongly invariant in  $G$ .

**SI6)** If  $H$  is strongly invariant in  $G$ , then in any finite direct power of  $G^n$ , the corresponding direct power  $H^n$  is strongly invariant.

To this list we can add the following properties

**SI7)** If  $H \leq K \leq G$  and  $H$  is strongly invariant in a group  $G$ , then  $K$  might not be strongly invariant in  $G$  (take the quaternion group  $Q_8$ ),

**SI8)** If  $H \leq K \leq G$  and  $K$  is strongly invariant in a group  $G$ , then  $H$  might not be strongly invariant in  $G$  (in  $\mathbf{Z}_2 \oplus \mathbf{Z}_4$  take the socle and a subsocle),

**SI9)** If  $H \leq K \leq G$  and  $K$  is strongly invariant in a group  $G$ , then  $K/H$  might not be strongly invariant in  $G/H$  (again in  $G = H \oplus K = \mathbf{Z}_2 \oplus \mathbf{Z}_4$ , the socle  $H + 2K$  is strongly invariant in  $G$  but  $(H + 2K)/2K = \mathbf{Z}_2$  is not strongly invariant in  $G/2K = \mathbf{Z}_2 \oplus \mathbf{Z}_2$  (see diagram on the fourth page of this article).

We add here an important property

**PROPOSITION 1.** *Fully invariant direct factors are strongly invariant.*

*Proof.* Let  $H$  be a fully invariant direct factor of  $G$ , that is,  $G = HK$ ,  $H \cap K = \{1\}$ , and let  $f : H \rightarrow G$  be any homomorphism. By hypothesis the composition  $f \circ p_H : G \rightarrow G$  (here  $p_H$  denotes the projection), maps  $H$  into  $H$ . Hence so does  $f$ .  $\square$

Hence

$$\{\text{fully inv. direct factors}\} \subseteq \{\text{strongly inv.}\} \subseteq \{\text{fully inv.}\}$$

In this paper we study the strongly invariant subgroups of Abelian groups and determine two extreme classes of Abelian groups related to this notion. Finally, some open questions are stated.

For notions, notations and results in the noncommutative case we refer to [8] and [9], and for those in the Abelian case, to [2] and [3]. To simplify the writing, we abbreviate strongly invariant by s-i.

**2. Preliminary results and examples.** The following characterisation (ii) will be useful

LEMMA 2. *The following conditions are equivalent*

- (i)  *$N$  is s-i in  $G$ ,*
- (ii) *for any subgroup  $H$  of  $G$ ,  $H \leq N$  whenever a subgroup epimorphism  $N \rightarrow H$  exists,*
- (iii) *for every group morphism  $f : N \rightarrow G$  the composition with the projection  $p_N : G \rightarrow G/N$  is the trivial homomorphism.*

COROLLARY 3. *Let  $N$  be a proper subgroup of a group  $G$  and  $x \in G - N$ . If there exists an epimorphism  $N \rightarrow \langle x \rangle$  then  $N$  is not s-i.*

We already know – Proposition 1 – that fully invariant direct factors are s-i and, both fully invariant and direct factor are, separately transitive properties. That’s why we can combine these properties

LEMMA 4. *If  $T \leq H \leq G$  with  $T$  fully invariant direct factor of  $H$  and  $H$  s-i in  $G$  then  $T$  is s-i in  $G$ .*

*Proof.* Let  $f : T \rightarrow G$  be a group morphism and  $H = T \times L$ . Since  $H$  is s-i in  $G$ , the composition  $f \circ p_T^H : H \rightarrow G$  maps  $H$  into  $H$ . Thus, as endomorphism of  $H$ ,  $(f \circ p_T^H)(T) \leq T$  (because  $T$  is fully invariant) and so  $f(T) \leq T$ . □

Notice that this result also shows that the class of all s-i subgroups, is *closed under fully invariant direct factors*.

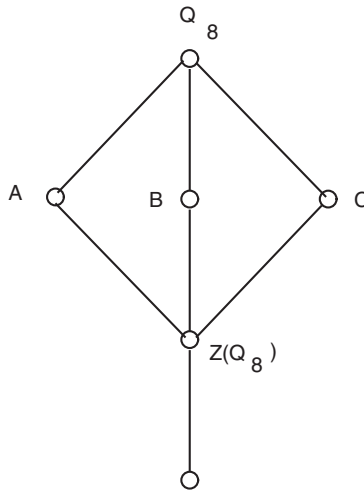
The second possible combination would be

*If  $T \leq H \leq G$  with  $T$  s-i in  $H$  and  $H$  fully invariant direct factor in  $G$  then  $T$  is s-i in  $G$ .*

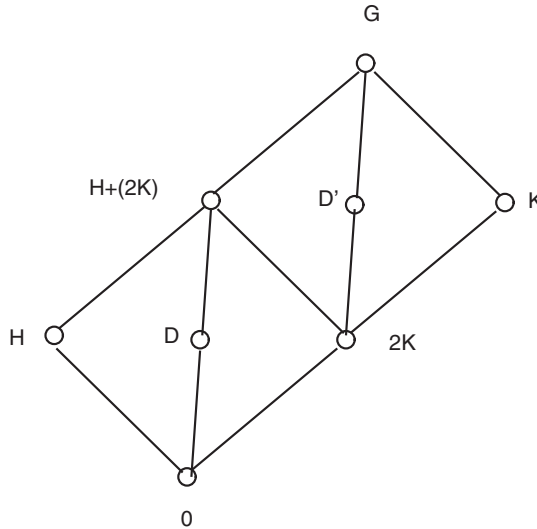
*However this fails:* to see this consider  $G = H \oplus L = \mathbf{Z}(2^\infty) \oplus \mathbf{Z}(2)$  with  $K = \mathbf{Z}(2) < H$ . Then  $K = S(H)$ , the socle, is s-i in  $H$ . On the contrary, it is not s-i in  $G$ , since the composition of the isomorphism  $K \cong L$  with the injection  $i_L : L \rightarrow G$  does not map  $K$  into  $K$ . Finally, since  $\text{Hom}(H, L) = \text{Hom}(\mathbf{Z}(2^\infty), \mathbf{Z}(2)) = 0$ , for any homomorphism  $H \xrightarrow{f} G$ , the projection on the second factor  $L = \mathbf{Z}(2)$ , is trivial. Hence  $f(H) \leq H$  and so  $H$  is s-i in  $G$ .

This Abelian example can be also used to prove **SI1**.

**2.1. Diagram examples.** In the quaternion group  $Q_8$ , the center is s-i but none of the (cyclic) subgroups  $A$ ,  $B$  or  $C$  is.



In  $G = H \oplus K = \mathbf{Z}_2 \oplus \mathbf{Z}_4$  the subgroup  $2K$  is not s-i but  $H + 2K$  is.



From the already mentioned Internet reference [10], here are some *examples* of s-i subgroups: the normal Sylow subgroups, the normal Hall subgroups, subgroups defined as the subgroup generated by elements of specific orders (for Abelian groups these are denoted  $G[n]$ ), the omega subgroups (i.e.,  $\Omega_j(G) = \{x \in G \mid x^{p^j} = 1\}$ ) of a group of prime power order and the perfect core of a group, and, some *non-examples*: the derived subgroup (two element subgroup) of  $M_{16}$  (i.e.,  $\langle a, x \mid a^8 = x^2 = 1, xax = a^5 \rangle$  - see diagram on the eleventh page of this article), the normal Klein four-subgroup of the symmetric group  $S_4$ , and  $S_2$  in  $S_3$ . The only Abelian nonexample given is  $\mathbf{Z}_2$  in  $V_4 = \mathbf{Z}_2 \oplus \mathbf{Z}_2$  (the Klein group).

We may also add some fully invariant nonexamples (which clearly are also s-i nonexamples [10]):  $A_3$  in  $A_4$ , or  $A_5$ , or  $S_4$ ;  $A_4$  in  $A_5$ ;  $S_2$  in  $S_3$  or in  $S_4$ .

**Remarks.**

- (1) It is well-known that  $N$  is normal and maximal in  $G$  whenever  $|G : N| = 2$ . However, there are subgroups of index two which are not s-i: take  $A, B$  or  $C$  in  $Q_8$  (see previous diagram).
- (2) Let  $A$  be any group and  $B$  a subgroup of  $C$  with  $A \cong B$ . Since the endomorphism  $A \times C \xrightarrow{p_A} A \xrightarrow{\cong} B \xrightarrow{\cong} \{1\} \times B \longrightarrow A \times C$  does not map  $A$  into  $A$ ,  $A$  is not fully invariant (and so nor s-i) in  $A \times C$ . Hence *there are not (nontrivial) groups which are fully invariant in every containing group*. Same for s-i.

**3. S-i subgroups of Abelian groups.**

**3.1. Preliminaries.** For easy reference, we first mention two results (the second is SI3)

LEMMA 5. *In any group  $G$ , for any positive integer  $n$ , the subgroup  $G[n]$  is s-i in  $G$ .*

However, in a  $p$ -group, any proper subocle is not s-i. Indeed, for a proper subocle an epimorphism onto an ‘outside’ direct summand can be constructed.

LEMMA 6. *Sums of s-i subgroups are also s-i.*

Further, since s-i subgroups are fully invariant we adapt a well-known result (Lemma 9.3, [2]) as follows

PROPOSITION 7. *Let  $N$  be a s-i subgroup of a group  $G = H \oplus K$ . Then  $N = (N \cap H) \oplus (N \cap K)$  and  $N \cap H, N \cap K$  are s-i in  $H$  and  $K$  respectively. Conversely, if  $H_1$  and  $K_1$  are s-i subgroups of  $H$  and  $K$  respectively, then  $H_1 \oplus K_1$  is s-i in  $G$  if and only if for every  $\delta : H_1 \longrightarrow K, \delta(H_1) \leq K_1$  and for every  $\gamma : K_1 \longrightarrow H, \gamma(K_1) \leq H_1$ .*

*Proof.* In order to check that  $N \cap H$  is strongly invariant in  $H$ , let  $f : N \cap H \longrightarrow H$  be a group morphism. This can be extended to  $N$  with zero on  $N \cap K$  and so gives a group morphism  $\bar{f} : N \longrightarrow G$ . Since  $N$  is strongly invariant in  $G, \bar{f}(N) \leq N$  and so  $f(N \cap H) \leq N$ . Hence  $f(N \cap H) \leq N \cap H$ . For the converse, just notice that every morphism  $f : H_1 \oplus K_1 \longrightarrow G$  is determined by a  $2 \times 2$  matrix  $\begin{bmatrix} \alpha & \gamma \\ \delta & \beta \end{bmatrix}$  with  $\alpha : H_1 \longrightarrow H, \gamma : K_1 \longrightarrow H, \delta : H_1 \longrightarrow K$  and  $\beta : K_1 \longrightarrow K$  and by hypothesis  $\alpha(H_1) \leq H_1$  and  $\beta(K_1) \leq K_1$ . Hence  $f(H_1 \oplus K_1) \leq H_1 \oplus K_1$  if and only if for every  $\delta : H_1 \longrightarrow K, \delta(H_1) \leq K_1$  and for every  $\gamma : K_1 \longrightarrow H, \gamma(K_1) \leq H_1$ . □

These necessary and sufficient conditions can be simplified using the following

**Definition.** For any pair of groups  $G, H$ , the  $G$ -socle of  $H$  is  $S_G(H) = \sum \{imf | f : G \longrightarrow H\}$ .

Using it,  $H_1 \oplus K_1$  is s-i in  $G$  if and only if  $S_{H_1}(K) \leq K_1$  and  $S_{K_1}(H) \leq H_1$ .

Therefore in order to find the s-i subgroups of a direct sum we need to find the s-i subgroups of the components, and after this, select among these accordingly.

As special cases,  $H_1 \oplus K_1$  is s-i in  $G$  if both  $H_1, K_1$  are s-i in  $G$  (use Lemma 6), or if  $\text{Hom}(H_1, K) = 0 = \text{Hom}(K_1, H)$ .

In [9], subgroups of a direct sum are classified as direct sums of subgroups and diagonals. The previous Proposition shows that the diagonals are not  $s$ -i subgroups.

Using early results of Kaplansky [4], we can easily dispose of  $s$ -i subgroups of divisible groups.

PROPOSITION 8.

- (a) *Torsion-free divisible groups are  $s$ -i simple.*
- (b) *The only  $s$ -i subgroups of a divisible  $p$ -group  $G$  are the subgroups  $G[p^n]$  for positive integers  $n$ .*
- (c) *The  $s$ -i subgroups of a divisible group  $G$  are  $G$  itself and the  $s$ -i subgroups of its torsion part  $T(G)$ , i.e. (see Proposition 11 below), direct sums of  $G[p^n]$  for different prime numbers  $p$  and positive integers  $n$ .*

Actually, in [4] (exercises 66–69), similar results are proven for characteristic (and fully invariant) submodules of modules over principal rings.

As for (b) notice that since divisible  $p$ -groups are direct sums of  $\mathbf{Z}(p^\infty)$ , and since the functorial subgroup  $G[n]$  commutes with direct sums, the  $s$ -i subgroups of a divisible  $p$ -group are direct sums of copies of the same  $\mathbf{Z}(p^n)$ .

Using Proposition 7, we reduce the determination of  $s$ -i subgroups to reduced groups

COROLLARY 9. *Let  $G = D(G) \oplus R$  be a decomposition of a group  $G$  with  $D(G)$  its divisible part and  $R$  a reduced group. Every  $s$ -i subgroup  $N$  of  $G$  has the form  $N = D_1 \oplus R_1$  with  $D_1$   $s$ -i subgroup in  $D(G)$ ,  $R_1$   $s$ -i subgroup in  $R$ . Conversely, a direct sum  $D_1 \oplus R_1$  with  $D_1$   $s$ -i subgroup in  $D(G)$ ,  $R_1$   $s$ -i subgroup in  $R$  is a  $s$ -i subgroup in  $G$  if and only if  $S_{D_1}(R) \leq R_1$  and  $S_{R_1}(D(G)) \leq D_1$ .*

Since  $s$ -i subgroups of divisible groups are listed above, and  $G$ -socles commute with direct sums, we finally obtain

COROLLARY 10. *A direct sum  $D_1 \oplus R_1 \neq G$  with  $D_1$   $s$ -i subgroup in  $D(G)$ ,  $R_1$   $s$ -i subgroup in  $R$  is a  $s$ -i subgroup in  $G$  if and only if  $S_{R_1}(\mathbf{Q}) = 0$  and, for every prime number  $p$ , if  $D_1 = \bigoplus \mathbf{Z}(p^n)$ , then  $S_{R_1}(\mathbf{Z}(p^\infty)) \leq \mathbf{Z}(p^n)$  and  $S_{D_1}(R) \leq R_1$ .*

**3.2. Torsion groups.** First of all, if  $A$  is a subgroup of a torsion group  $G$ , then both  $A$  and  $G$  decompose into  $p$ -components.

PROPOSITION 11. *Let  $A$  be a subgroup in a torsion group  $G$ . Then  $A$  is  $s$ -i in  $G$  if and only if for every prime  $p$ ,  $A_p$  is  $s$ -i in  $G_p$ .*

*Proof.* For an arbitrary prime  $p$ , let  $A_p \xrightarrow{\delta_p} G_p$  be a group morphism. Since  $A_p$  is a direct summand, we trivially extend this to an  $A \rightarrow G_p$  and further to  $A \rightarrow G$ , so that  $\delta_p(A_p) \leq A$ . Since  $\delta_p(A_p) \leq G_p$  we also have  $\delta_p(A_p) \leq A \cap G_p = A_p$ . Conversely, let  $A \xrightarrow{\delta} G$  be a group morphism. Since we can decompose it into  $A = \bigoplus_p A_p \xrightarrow{\oplus \delta_p} \bigoplus_p G_p = G$  and by hypothesis,  $\delta_p(A_p) \leq A_p$ , finally,  $\delta(A) \leq A$ .  $\square$

This is similar to: fully invariant (or characteristic) subgroups of torsion groups are direct sums of fully invariant (characteristic)  $p$ -subgroups.

So the problem of determining the *s*-i subgroups in a torsion group reduces to *p*-groups. Further, using Corollary 7, the study reduces to reduced *p*-groups.

While an elaborate theory was needed in order to characterise (Kaplansky) fully invariant subgroups in special (fully transitive) classes of *p*-groups, determining the *s*-i subgroups of a reduced *p*-group is far more easy.

**PROPOSITION 12.** *The only (reduced) s-i subgroups of a reduced p-group are the subgroups  $G[p^n]$  for all positive integers  $n$ .*

*Proof.* One way is just Lemma 5. Conversely, suppose a subgroup  $N$  of a *p*-group  $G$  has not the form  $G[p^n]$ . Then two cases are possible.

**Case 1.**  $N$  is bounded (effectively) by  $p^n$  (i.e.,  $\max_{a \in N}(\text{orda}) = p^n$ ) but  $G[p^n] \not\subseteq N$ . In this case  $N$  is a direct sum of cyclic *p*-groups, bounded by  $p^n$  and at least one summand has order  $p^n$ . More, there exists an element  $x \in G - N$  of order  $p^n$ . Denote  $C$  a direct summand of order  $p^n$  and let  $N = C \oplus (\bigoplus_{i \in I} C_i)$ . Define  $f : N \rightarrow G$  by the zero morphism  $\bigoplus_{i \in I} C_i \rightarrow G$  and the isomorphism  $C \rightarrow \langle x \rangle \rightarrow G$  followed by inclusion. Then  $x \in f(N)$  and so  $f(N) \not\subseteq N$ . Hence the subgroup  $N$  is not a *s*-i subgroup.

**Case 2.**  $N$  is unbounded. Using the early (fundamental) Kulikov theorems, we infer that if a reduced *p*-group has elements of arbitrary high orders, then it also has cyclic direct summands of arbitrary high orders. Let  $N$  be reduced and let  $x \in G - N$ . In  $N$  we choose a direct summand  $C$  of order  $\geq \text{ord}(x)$ . More precisely, let  $|C| = p^n$  and let  $\text{ord}(x) = p^m$  with  $n \geq m$ . Since cyclic torsion groups are quasi-injective, we can extend the isomorphism  $p^{n-m}C \rightarrow \langle x \rangle \rightarrow G$  followed by inclusion, to a group morphism  $f : C \rightarrow G$  which still has  $x$  in its image. Finally if  $N = C \oplus E$ , the previous morphism and the zero morphism  $E \rightarrow G$  define a morphism  $\bar{f} : N \rightarrow G$  with  $x \in \bar{f}(N) = f(N)$ . Again, the subgroup  $N$  is not a *s*-i. □

**3.3. Torsion-free groups.** Here too, using Corollary 9, we deal only with reduced (torsion-free) groups. Since we already noticed that rank 1 torsion-free groups have no (proper) *s*-i subgroups, in the sequel we consider *only groups of rank 2 or more*.

A slight generalisation in the negative direction is the following

**PROPOSITION 13.** *Suppose  $A$  is a subgroup in a reduced torsion-free group  $G$  and  $r(A) < r(G)$ . If  $A$  contains a free direct summand  $F$  then  $A$  is not *s*-i.*

*Proof.* Owing to the rank inequality, there is an element  $b \in G - A$  with  $A \cap \langle b \rangle = 0$  and so  $\langle b \rangle \not\subseteq A$ . Since we can construct epimorphisms  $F \rightarrow \mathbf{Z} = \langle b \rangle$ , and extend them to epimorphisms  $A \rightarrow \langle b \rangle$ , the statement follows using Corollary 3. □

**COROLLARY 14.** *Torsion-free groups ( $r(G) \geq 2$ ) have no cyclic s-i subgroups.*

**COROLLARY 15.** *Torsion-free groups ( $r(G) \geq 2$ ) have no rational s-i subgroups.*

Since for  $\text{Hom}(A, \mathbf{Z})$  with torsion-free  $A$ ,  $\text{rk}(\text{Hom}(A, \mathbf{Z})) \geq n$  if and only if  $A$  has a free direct summand of rank  $n$  (Lewis [7]), we can rephrase the previous Proposition as follows

**PROPOSITION 16.** *In any reduced torsion-free group  $G$ , subgroups  $A$  with  $r(A) < r(G)$  and  $\text{Hom}(A, \mathbf{Z}) \neq 0$  are not *s*-i.*

Since in Proposition 26, very large classes of torsion-free groups will be found with no proper s-i subgroups, in what follows, we point out some positive results.

First recall that, in a nonhomogeneous torsion-free group  $G$ , the subgroups  $G(\mathbf{t})$  are pure and (proper) s-i. Hence, using Lemma 6, sums of subgroups  $G(\mathbf{t})$  are also s-i.

Since the subgroups  $G^*(\mathbf{t}) = \langle \{a \in G | \mathbf{t}(a) > \mathbf{t}\} \rangle$  are not always pure, we point out

**PROPOSITION 17.** *If the subgroup  $G^*(\mathbf{t})$  is pure then it is s-i.*

*Proof.* Let  $f : G^*(\mathbf{t}) \rightarrow G$  be a group morphism. Obviously we have  $f(\langle \{a \in G | \mathbf{t}(a) > \mathbf{t}\} \rangle) = \langle f(\{a \in G | \mathbf{t}(a) > \mathbf{t}\}) \rangle = \langle \{f(a) | a \in G : \mathbf{t}(a) > \mathbf{t}\} \rangle$ , so for  $f(G^*(\mathbf{t})) \leq G^*(\mathbf{t})$  we need  $\mathbf{t}_G(a) \leq \mathbf{t}_G(f(a))$ . Generally we only have  $\mathbf{t}_{G^*(\mathbf{t})}(a) \leq \mathbf{t}_G(f(a))$  and  $\mathbf{t}_{G^*(\mathbf{t})}(a) \leq \mathbf{t}_G(a)$ . If  $G^*(\mathbf{t})$  is pure, the inequality required follows because  $\mathbf{t}_{G^*(\mathbf{t})}(a) = \mathbf{t}_G(a)$ . □

**Example.** If  $G$  is separable then for every type  $\mathbf{t}$ ,  $G^*(\mathbf{t})$  is pure in  $G$  and so also s-i.

A nice Conjecture would be: *any torsion-free s-i subgroup is a sum of subgroups  $G(\mathbf{t})$ .* A study of such subgroups is not known (so far) to the author.

**3.4. Mixed groups.** First of all, we can dispose of torsion subgroups in mixed groups. Indeed

**PROPOSITION 18.** *Let  $N$  be a subgroup of a mixed group  $G$ . Then  $T(N)$  is s-i subgroup of  $T(G)$  if and only if  $T(N)$  is s-i subgroup of  $G$ .*

*Proof.* One way is **(SI2)**: if  $T(N)$  is s-i in  $G$ , it is also s-i in  $T(G)$ . Conversely, let  $f : T(N) \rightarrow G$  be a group morphism. Clearly,  $f(T(N)) \leq T(G)$  and so we can consider the morphism  $\tilde{f} : T(N) \rightarrow T(G)$  obtained from  $f$  by codomain restriction. By hypothesis  $\tilde{f}$  maps  $T(N)$  into  $T(N)$ . Hence so does  $f$ . □

**COROLLARY 19.** *Let  $N$  be a torsion subgroup of a mixed group  $G$ . Then  $N$  is s-i in  $G$  if and only if it is s-i in  $T(G)$ .*

**COROLLARY 20.** *The torsion s-i subgroups in a mixed group are only the subgroups  $G[n]$  for all positive integers  $n$ .*

Further, according to Proposition 7, in a splitting mixed group, the determination of the s-i subgroups reduces to the previous two subsections. Therefore infinite cyclic subgroups are not s-i. Actually, this can be proved for arbitrary mixed groups.

**LEMMA 21.** *No infinite cyclic subgroup is s-i in a (genuine) mixed group.*

*Proof.* Indeed, in a genuine mixed group, let  $a, b \in G$  with  $\text{ord}a = \infty$  and  $\text{ord}b = p$ . There is a canonical subgroup epimorphism  $\gamma : \langle a \rangle \rightarrow \langle b \rangle$  and so  $\langle a \rangle$  is not s-i. □

Again we can generalise this at once to

**PROPOSITION 22.** *If in a (genuine) mixed group a subgroup contains a free direct summand, it is not s-i.*

Finally, torsion-free s-i subgroups abound in mixed groups, splitting or not. More, one can construct lots of fully invariant torsion-free subgroups in the following ways.

Splitting groups  $G = T \oplus H$  with torsion-free  $p$ -divisible  $H$  for each prime  $p$  such that the  $p$ -component  $T_p$  is (nonzero) reduced, or, extensions of the type  $0 \rightarrow H \rightarrow$



$G \longrightarrow U \longrightarrow 0$  with torsion-free  $p$ -divisible  $H$ , for a given prime  $p$ , such that  $U$  is  $p$ -reduced (e.g., see [1]).

**4. Two extreme classes.** In this section our goal is to determine two extreme classes of Abelian groups: the  $s$ -i simple groups (i.e., groups without proper  $s$ -i subgroups) and the groups in which every subgroup is  $s$ -i.

Since

$$\text{strongly invariant} \implies \text{fully invariant} \implies \text{characteristic} \implies \text{normal}$$

we derive

$$\text{simple} \implies \text{charact. simple} \implies \text{fully inv. simple} \implies \text{strongly inv. simple.}$$

It is well-known (e.g., see [8]) that an arbitrary finite group – or even a group with a minimal normal subgroup – is characteristically simple if it is either simple or it is a direct product of isomorphic simple groups.

**Examples.**  $\mathbf{Z}(2) \oplus \mathbf{Z}(2)$  is  $s$ -i simple but  $\mathbf{Z}(2) \oplus \mathbf{Z}(4)$  is not.

From now on,  $G$  denotes an Abelian group. Notice that for a group endomorphism  $f : G \longrightarrow G$ , clearly  $f(nG) = nf(G) \leq nG$  and so  $nG$  is a fully invariant subgroup of  $G$ . However, for the subgroup  $nG$  of a group  $G$  and a group morphism  $f : nG \longrightarrow G$ , if  $nG \neq G$ , the equality  $f(nG) = nf(G)$  makes no sense because  $f$  is not defined on the whole group  $G$ . More

LEMMA 23. *In any torsion-free group  $G$ , a subgroup  $nG$  is  $s$ -i in  $G$  if and only if  $nG = G$ .*

*Proof.* Only one way needs justification. If  $nG \neq G$ , take the group morphism  $t_{\frac{1}{n}} : nG \longrightarrow G$  given by  $t_{\frac{1}{n}}(ng) = g$  for every  $g \in G$ . Then  $t_{\frac{1}{n}}(nG) = G \not\leq nG$  and so  $nG$  is not  $s$ -i. □

**4.1. Fully invariant simple Abelian groups.** Since the torsion part, the socle, the primary components and the divisible part of an Abelian group are fully invariant subgroups, we can easily dispose of *fully invariant-simple* Abelian groups.

PROPOSITION 24.

- (i) *No genuine mixed group is fully invariant simple.*
- (ii) *The only torsion groups which may be fully invariant simple are the  $p$ -groups.*
- (iii) *A  $p$ -group is fully invariant simple if and only if it is elementary.*
- (iv) *A torsion-free group  $G$  is fully invariant simple if and only if it is divisible, i.e., a direct sum of  $\mathbf{Q}$ .*

*Proof.* (iii) Indeed, in a nontrivial  $p$ -group, the socle  $G[p]$  is a nontrivial fully invariant subgroup. If  $G$  is fully invariant simple we must have  $G[p] = G$ , i.e.,  $G$  is elementary. Conversely, no proper subsocle is fully invariant, so elementary  $p$ -groups are fully invariant simple.

(iv) Indeed, all the subgroups  $\{nG | n \in \mathbf{Z}\}$  being fully invariant, we must have  $nG = G$  for each  $n \in \mathbf{N}^*$  and hence  $G$  is divisible, i.e., a direct sum of copies of  $\mathbf{Q}$ . Conversely, recall that a torsion-free group  $G$  is said to be *fully transitive* if for any two nonzero elements  $a, b$  with characteristics  $\chi(a) \leq \chi(b)$ , there exists an endomorphism

$f \in \text{End}(G)$  such that  $f(a) = b$ . It is not hard to prove that any homogeneous separable torsion-free group is fully transitive (see [5]).

Moreover (p.198 [5]), the fully invariant subgroups of a torsion-free group  $G$  coincide with the subgroups of the form  $nG$  ( $n \in \mathbf{Z}$ ) if and only if  $G$  is a homogeneous fully transitive group of idempotent type. The combination of these two results shows that any torsion-free divisible group is fully invariant simple.  $\square$

**Remarks.**

- (1) *An Abelian group is characteristically simple if and only if it is fully invariant simple.*

Indeed, from (iii) and (iv) above, we just have to notice that these classes are both vector spaces over  $\mathbf{Z}(p)$  and  $\mathbf{Q}$  respectively, and so, also characteristically simple.

- (2) *Actually more can be proved (see [4], exercise 66): any torsion-free divisible module over any principal ideal ring is characteristically simple.*

**4.2. S-i simple Abelian groups.** As first examples, both  $\mathbf{Z}$  and  $\mathbf{Q}$  are s-i simple (use Lemma 23 for the first).

On the contrary, in a cyclic group, every subgroup is fully invariant.

Again, since the socle, the torsion part, the divisible part and some special (pure) subgroups (in the torsion-free case) of a group are s-i, mostly all of the possible cases are easily covered.

**PROPOSITION 25.**

- (i) *Genuine mixed groups are not s-i simple.*
- (ii) *A torsion group is s-i simple if and only if it is an elementary  $p$ -group (i.e., if and only if it is fully invariant simple).*
- (iii) *Any s-i simple group is either divisible or reduced.*
- (iv) *Any s-i simple torsion-free group is homogeneous.*
- (v) *Any torsion-free divisible group is s-i simple.*

*Proof.*

- (i) For a genuine mixed group, i.e.,  $0 \neq T(G) \neq G$ , the torsion part  $T(G)$  is s-i.
- (ii) As in the fully invariant case, s-i simple torsion groups must be  $p$ -groups. Since the socle of a  $p$ -group is s-i (and  $0 \neq G[p]$ ), only  $p$ -groups with  $G = G[p]$ , that is, elementary  $p$ -groups, can be s-i simple. Finally, elementary  $p$ -groups are s-i simple because every proper subsocle is not s-i (e.g., use Lemma 2).  
An alternative proof can be given using Proposition 12.
- (iii) For a group  $G$ , the divisible part  $D(G)$  is a fully invariant direct summand. As such (see Lemma 1), it is also s-i. Thus, groups with  $0 \neq D(G) \neq G$  are not s-i simple. So only  $D(G) = G$  (divisible) or  $D(G) = 0$  (reduced) are possible.
- (iv) For a torsion-free group  $G$  and a given type  $\mathbf{t}$ , consider the fully invariant (pure) subgroup  $G(\mathbf{t}) = \{a \in G \mid \mathbf{t}(a) \geq \mathbf{t}\}$ . Since group morphisms do not decrease heights, it is readily seen that this subgroup is also s-i (but the purity of this subgroup is essential: heights, characteristics or types computed in  $G(\mathbf{t})$  are equal to those computed in  $G$ ). Since there is at least one type  $\mathbf{t}$  with  $G(\mathbf{t}) \neq 0$ , the s-i simple torsion-free groups must verify  $G = G(\mathbf{t})$ . Hence these are homogeneous.

- (v) These groups are fully invariant simple (see (iv) in the previous list) and so also s-i simple.

□

Therefore what remains is to single out the *reduced homogeneous torsion-free s-i simple groups*.

None of the tools above can be used in this case: proper  $nG$  are generally not s-i,  $G[n]$  are s-i but equal zero in the torsion-free case and  $G(\mathbf{t}) = G$  in the homogeneous case.

It is easy to check that *completely decomposable reduced homogeneous groups are also s-i simple*. But we can do better since there are large classes of fully transitive (torsion-free) groups which are s-i simple.

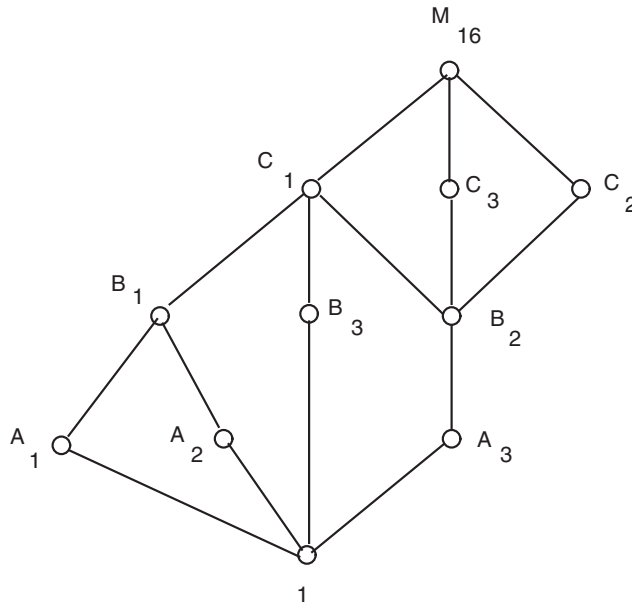
**PROPOSITION 26.** *Fully transitive (torsion-free) homogeneous groups of idempotent type are s-i simple.*

*Proof.* The claim follows since the fully invariant subgroups of a torsion-free group  $G$ , coincide with subgroups of the form  $nG$  ( $n \in \mathbf{Z}$ ) if and only if  $G$  is homogeneous fully transitive of idempotent type, and (see Lemma 23)  $nG$  is s-i in a torsion-free group  $G$  if and only if  $nG = G$ . □

Among fully transitive torsion-free groups we mention the separable and algebraic compact groups.

Since (so far) there is no structure theorem for fully transitive homogeneous groups of idempotent type, a structure theorem for reduced torsion-free s-i simple groups seems (so far) out of reach.

A noncommutative example: consider  $M_{16} = \langle a, x \mid a^8 = x^2 = 1, xax = a^5 \rangle$ . Here is its subgroup lattice



Clearly,  $A_1 \cong A_2 \cong A_3$  as order 2 cyclic subgroups. Suitable isomorphisms composed with inclusions, give epimorphisms as in Corollary 3. Since  $A_i \cap A_j = 1$

for every  $i, j \in \{1, 2, 3\}$ , none of these atoms is s-i. So are  $B_2$  and  $B_3$  (see Introduction), as order 4 cyclic subgroups, respectively  $C_2$  and  $C_3$ , as order 8 cyclic subgroups. Finally, take the projections  $B_1 \rightarrow A_2$  (Klein-cyclic) and  $C_1 \rightarrow B_2$  ( $\mathbf{Z}_2 \oplus \mathbf{Z}_4 \rightarrow \mathbf{Z}_4$ ). These are clearly epimorphisms and again Corollary 3 shows these are not s-i. Hence,  $M_{16}$  is s-i simple.

**4.3. Groups in which every subgroup is s-i.** Using the denial of Lemma 2, if the subgroup lattice  $L(G)$  has two isomorphic atoms, none of them is s-i in  $G$ . Hence

PROPOSITION 27. *In a  $p$ -group  $G$  every subgroup is s-i if and only if  $G$  is cocyclic.*

*Proof.* Since the subgroup lattice of a cocyclic group is a chain, the condition is sufficient. It is also necessary because a  $p$ -group is cocyclic if and only if it has a unique smallest nonzero subgroup. (An alternative proof can be given using Proposition 8).  $\square$

Further

PROPOSITION 28. *In a torsion group every subgroup is s-i if and only if each  $p$ -component has this property.*

*Proof.* By **SI2**, the condition is necessary. Conversely, let  $H$  be a subgroup of a torsion group  $G$  and let  $H = \bigoplus_p H_p \leq \bigoplus_p G_p = G$  be the primary component decompositions. A group morphism  $f : H \rightarrow G$  decomposes as  $f = \bigoplus_p f_p : \bigoplus_p H_p \rightarrow \bigoplus_p G_p$  with group morphisms  $f_p : H_p \rightarrow G_p$ . By hypothesis  $f_p(H_p) \leq H_p$  and so  $f(H) = f(\bigoplus_p H_p) = \bigoplus_p f_p(H_p) \leq \bigoplus_p H_p = H$  because  $\text{Hom}(H_p, H_q) = 0$  for different prime numbers  $p, q$ .  $\square$

Finally,

PROPOSITION 29. *There are no nonzero torsion-free nor genuine mixed groups with the above property.*

*Proof.* Indeed, let  $G$  be a torsion-free group. If the rank of  $G$  is at least two, pick  $a, b \in G$  with  $\text{ord}a = \text{ord}b = \infty$  and  $\langle a \rangle \cap \langle b \rangle = 0$ . According to (the denial of) Lemma 2, none of  $\langle a \rangle, \langle b \rangle$  is s-i. In the remaining case, we already proved (previous subsection) that rank one torsion-free groups are s-i simple.

Now let  $G$  be a genuine mixed group. We can find elements  $a, b \in G$  with  $\text{ord}a = \infty$ ,  $\text{ord}b = p$  (for a suitable prime number  $p$ ) and a nontrivial epimorphism  $\langle a \rangle \rightarrow \langle b \rangle$ . Hence, by Corollary 3,  $\langle a \rangle$  is not s-i.  $\square$

Therefore

COROLLARY 30. *The only Abelian groups in which every subgroup is s-i, are the direct sums of cocyclic groups, at most one for each prime number.*

## 5. Open questions.

- (1) Since fully invariant direct summands are strongly invariant, we may wonder whether *fully invariant pure* (called pfi in [5], see. p. 36) *subgroups are also s-i*. If this is not true, we may wonder whether *fully invariant balanced* subgroups are s-i.

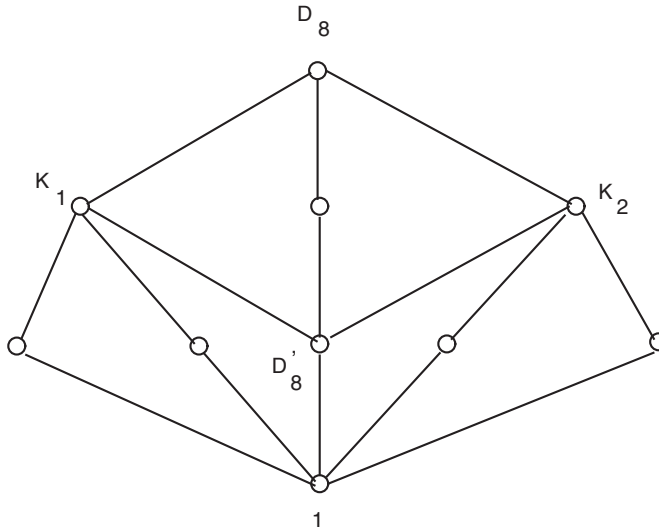
For instance, in a torsion-free group  $G$ , for a type  $\mathbf{t}$ ,  $G(\mathbf{t})$  is fully invariant pure (generally not direct summand) and strongly invariant.

- (2) Since (see **FI2**) fully invariant subgroups have not the intermediate subgroup property, subgroups that are fully invariant in every intermediate subgroup, may be considered and studied.

Since s-i subgroups share the intermediate subgroup property (i.e., **SI2**), for subgroups in an arbitrary group we have

$$\text{s-i} \implies \text{interm. fully invariant-subgroup} \implies \text{fully invariant.}$$

*Examples* show these classes are different. Indeed, in the above given diagram, for  $G = M_{16}$  take the Klein subgroup  $H = B_1$ . This is not s-i in  $G$  but is fully invariant in  $C_1$  (Klein is fully invariant in  $\mathbf{Z}_2 \oplus \mathbf{Z}_4$ ) and in  $G$ . As for the second (proper) inclusion, take the commutator  $D'_8 = \langle s^2 \rangle$  which is fully invariant in the dihedral group  $D_8 = \langle s, t; s^4 = t^2 = 1, sts = t \rangle$  but not fully invariant in any of the Klein intermediate subgroups ( $K_1$  and  $K_2$  in the following diagram).



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