A SQUARE-ZERO MATRIX WITH NONZERO DETERMINANT

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Abstract. If a 2×2 diagonal matrix has this property, the diagonal entries, say a, b must satisfy $a^2 = b^2 = 0$ but $ab \neq 0$. Two examples are given over not commutative rings and over commutative rings.

Problem. Find a square-zero 2×2 *diagonal* matrix with nonzero determinant over a ring R, that is, $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with $a^2 = b^2 = 0$ but $ab \neq 0$.

Solution. 1) Over a **not** commutative ring: take $R = \mathbb{M}_2(\mathbb{Z}_2)$ and $E_{12}^2 = E_{21}^2 = 0$, $E_{12}E_{21} = E_{11} \neq 0 \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = E_{12} + E_{21}$.

2) Over the commutative rings $\mathbb{Z}_n$, this is **not** possible: $\overline{a} \in N(\mathbb{Z}_n)$ for $n = p_1^{k_1}...p_m^{k_m}$, iff $p_1...p_m \mid a$, but $\overline{a} \in S(\mathbb{Z}_n)$ iff $p_1^{[\frac{k_1}{2}]+1}...p_m^{[\frac{k_m}{2}]+1} \mid a$. Hence for $\overline{a}, \overline{b} \in S(\mathbb{Z}_n)$, $\overline{ab} = \overline{0}$ since $n \mid ab$ [here $N(R)$ denotes the nilpotent elements of R and $S(R)$ its square-zero elements].

3) Over some suitable commutative ring.
From [1]:
Problem E 1665. Construct a commutative ring in which the square of each element is zero but not every product is zero. Prove that such a ring must have at least 8 elements.

Solution. The set of polynomials $2a + bx$, mod $(4, x^2)$ where $a, b \in \mathbb{Z}$, furnishes an example of such a ring. Clearly $(2a+x)^2 = 4a^2+4ax+x^2 \equiv 0$ and $(2+x)x \not\equiv 0$. Suppose the ring has the properties and let $ab \neq 0$. Then $a + b \neq 0$; further $ab \neq a$ or b since $ab = a$ implies $0 = ab^2 = ab$. Thus $0, a, b, a + b, ab$ are distinct. Next $ab + a, ab + b, ab + a + b$ are also distinct from these.
Notice that this ring has no identity.
The subject of "square-zero rings" is elaborated in [2].

4) Over some suitable commutative ring with identity.
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$\mathbb{Z}_4[X]/(X^2)$. The elements of this ring are represented by polynomials of degree at most 1 over $\mathbb{Z}_4$. Then $2 + (X^2)$ and $X + (X^2)$ have the required properties.

References

[1] L. Carlitz *Solution to E 1665*. Amer. Math. Monthly **72** (1965), 80.
[2] R. P. Stanley *Zero square rings*. Pacific J. Math. **30** (1969), 811-824.