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Grigore Călugăreanu*

Singular matrices that are products of two idempotents or products of two nilpotents

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Abstract: Over commutative domains we characterize the singular 2×2 matrices which are products of two idempotents or products of two nilpotents. The relevant casees are the matrices with zero second row and the singular matrices with only nonzero entries.

Keywords: idempotent matrix, nilpotent matrix, quadratic Diophantine equation, 2×2 matrix

MSC: 15B33, 15B99, 11D09, 11T99

1 Introduction

The question of presenting non-units of an algebra as a product of idempotents seems to have begun by the work of Howie [9] (1966) who showed that any non-injective mapping from a finite set into itself is a product of idempotents.

In view of the analogy between the theories of transformations of finite sets and linear transformations of finite dimensional vector spaces, J. A. Erdos [8] (1967), proved that every singular $n \times n$ matrix with entries in a *field* K can be expressed as a product of idempotents over K. Then C. S. Ballantine [4] (1978), quantified this result by relating the minimum number of idempotents required to the rank of the matrix: the $n \times n$ matrix A is the product of k idempotent matrices if and only if $\text{rank}(I_n - A) \leq k\text{nullity}(A)$, and in particular proved that any $n \times n$ matrix can be written as the product of n idempotents over the field K. T. T. J. Laffey [11] (1983) proved that this decomposition into idempotents holds for singular matrices over division rings and commutative Euclidean domains. We quote from the final remarks: "The proof of Lemma 2 (any zero second row 2×2 matrix over a commutative Euclidean ring R can be expressed as a product of idempotents in $\mathbb{M}_2(R)$) does not lead to a bound on the number of idempotents required. We do not know if such a bound exists".

We also mention [12], in which it is shown that given any positive integer N, there exists an integral 2×2 matrix A which is the product of N but no fewer idempotents in $\mathbb{M}_2(\mathbb{Z})$, and there exists a matrix $B \in \mathbb{M}_2(\mathbb{Z})$ which is the product of N but no fewer nilpotent matrices in $\mathbb{M}_2(\mathbb{Z})$.

Many improvements (and generalizations) of these results were made in the last 30 years or so, many of these in the last 6-7 years.

Recently, [10] (2019) provides a nice survey of the progress that has been made on this long-standing problem. It includes an exhaustive bibliography of the subject.

As the reviewer pointed out, there are two more recent papers not mentioned in [10], somewhat related to the subject, an interested reader could consult.

In [6], a conjecture the paper focuses upon is equivalent to "A non-Euclidean PID has a 2×2 singular matrix which is not a product of idempotent matrices" and in [7], among other things, a domain R is said to satisfy property (ID2) if every 2×2 singular matrix over R is a product of idempotent matrices. The authors

***Corresponding Author: Grigore Călugăreanu:** Babeş-Bolyai University, Cluj-Napoca, Romania, E-mail: calu@math.ubbcluj.ro

prove that if p and q are two elements of R with some conditions on the degree and roots of p and q (p and q are rational functions), the matrix $M = \begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix}$ is a product of idempotent matrices.

This note was prompted by the Laffey's remark mentioned above and by a remark made in [2]: over the integers, the matrix $A = \begin{bmatrix} 14 & 8 \\ 0 & 0 \end{bmatrix}$ decomposes into three or four idempotents, that is, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -7 & -4 \\ 14 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} -7 & -4 \\ 14 & 8 \end{bmatrix}$, "and *it can be shown* that this is in fact "a shortest" factorization for A." This seems to be the only example in the literature, of an integral matrix which is *not* a product of two idempotents.

Since, over any commutative ring, any product of finitely many idempotent (or nilpotent) matrices has zero determinant, in this note we address a converse: *determine the zero determinant* 2×2 *matrices which are products of two idempotents, or, products of two nilpotents*.

To simplify the writing we use the following

Definition. An element of a ring has the property *2I* (or *2N*) if it is a product of two idempotents (resp. two nilpotents).

Idempotents trivially have the property 2I ($e = 1 \cdot e = e \cdot e$) and 1 is the only unit which decomposes as product of idempotents. However, it is easy to find, over any (commutative) ring R, (nonzero) nilpotent 2×2 matrices which have *not* the 2N property.

In the second section, we first recall some decompositions over arbitrary rings and some elementary results related to such decompositions.

In the third section we characterize the 2×2 matrices which have the property 2I, while in the fourth section we characterize the 2×2 matrices which have the property 2N.

In the final section we discuss the 3×3 case and state an open question.

All characterizations amount to quadratic polynomial equations in two indeterminates over commutative domains, in the simple hyperbolic case.

Our results are:

Theorem 5. *Let R be a commutative domain. The matrix* $A = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$ *has property 2I if and only if* $\alpha = 0$ *and/or* $\beta = 0$, *or else, there exists* $b \in R$ *such that* $\beta - b\alpha \neq 0$ *divides* $\alpha(1-\alpha)$. *Equivalently, the quadratic polynomial equation*

$$\alpha bz - \beta z + \alpha(1-\alpha) = 0$$

in the unknowns b, z *has solutions in* R. *In particular, this holds if* $\alpha \neq 0$ *divides* β *or else,* $\beta \neq 0$ *divides* α *or* $\alpha(1-\alpha)$.

Theorem 6 *Let* $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ *be a singular matrix with nonzero entries over a commutative domain* R. *The matrix A has property 2I if and only if the quadratic polynomial equation*

$$(\alpha+\delta)ax - \alpha(a+x) = \alpha(\delta-1)$$

in the unknowns a, x *has at least one solution* (a, x) *for which* α *divides* βx, β *divides* $\alpha(1-x)$, γ *divides* $\alpha(1-a)$ *and* α *divides* γa.

In particular, this holds in any of the following cases:

(i) β *divides* $\alpha(1-\alpha)$ *and* α *divides* γ,

(ii) α *divides* β *and* γ *divides* $\alpha(1-\alpha)$,

(iii) α *divides* $\beta(1-\gamma)$ *and* γ *divides* α,

(iv) β *divides* α *and* α *divides* $\gamma(1-\delta)$.

As for the 2N property, our results are:

Theorem 7. *Let R be a commutative domain. The matrix* $A = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$ *has property 2N if and only if* $\beta = 0$ *or* $\alpha, \beta \neq 0$ *and* α *divides* β^2.

Theorem 8. *Let* $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ *be a singular matrix with nonzero entries over a commutative domain* R. *The matrix* A *has property 2N if and only if* $\alpha + \delta$ *divides* $\alpha\delta$ *and the quadratic polynomial equation*

$$(\alpha + \delta)ax = \alpha\delta$$

in the unknowns a, x *has at least one solution* (a, x) *for which* α *divides* βx, β *divides* αx, γ *divides* αa *and* α *divides* γa.

For any ring R, $U(R)$ denotes the set of all units of R and E_{ij} denotes the $n \times n$ matrix with only zero entries, excepting the (i, j)-entry, which is 1. A commutative domain R is called *GCD* if for each pair $a, b \in R$, the greatest common divisor $\gcd(a, b)$ exists. Examples of GCD domains include unique factorization domains, principal ideal domains, Euclidean domains and fields.

2 Over rings

First recall that the standard form of a nontrivial 2×2 idempotent (resp. nilpotent) matrix over a domain is $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ with $a(1-a) = bc$ (resp. $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ with $a^2 + bc = 0$).

Secondly, we record some useful results which hold over any (unital) ring.

Lemma 1. *(i) In any ring, the properties 2I and 2N are invariant to conjugations.*
(ii) For (square) matrices, the properties 2I and 2N are invariant to transpose.

Lemma 2. *Over any ring,* 2×2 *matrices with three zero entries have property 2I. The matrices* aE_{11}, aE_{22} *have the 2N property but* aE_{12}, aE_{21} $(a \neq 0)$ *have not.*

Proof. As for 2I, notice that $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a-1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a \\ 0 & 1 \end{bmatrix}$. The other two possibilities follow by transpose or conjugation with the unit $E_{12} + E_{21}$. The matrices $aE_{11} = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $aE_{22} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}$, have the 2N property, but (simple computations show that) aE_{12}, aE_{21} $(a \neq 0)$ do *not* decompose in products of two nilpotents. □

Since every nilpotent 2×2 matrix over a GCD domain is similar to some multiple of E_{12}, it follows that

Corollary 3. *Nilpotent* 2×2 *matrices over GCD domains have property 2I but have not the property 2N (if nonzero).*

Over commutative domains, suppose a zero determinant 2×2 matrix has a zero entry. Then it has (at least) another zero entry, on the same row, or on the same column. So (by transpose and/or conjugation) for the property 2I (or 2N) it suffices to discuss the (zero determinant) *matrices with second zero row*. Since matrices with three nonzero entries have nonzero determinant, the only case left are the zero determinant *matrices with only nonzero entries*. This is done in the next section for 2I (and in section four for 2N).

3 Matrices with 2I

For a matrix A in the general case, with respect to 2I, we have to solve a system, denoted (SI), which we consider over a commutative domain R, E, F are nontrivial idempotents and $\det(A) = 0$. That is

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = EF = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} \begin{bmatrix} x & y \\ z & 1-x \end{bmatrix} =$$

$= \begin{bmatrix} ax+bz & ay+b(1-x) \\ cx+(1-a)z & cy+(1-a)(1-x) \end{bmatrix}$ with $\alpha\delta = \beta\gamma$, $a(1-a) = bc$ and $x(1-x) = yz$. This amounts to

$$\begin{gathered} ax+bz=\alpha \quad (1) \\ ay+b(1-x)=\beta \quad (2) \\ cx+(1-a)z=\gamma \quad (3) \\ cy+(1-a)(1-x)=\delta \quad (4) \\ a(1-a)=bc \quad (5) \\ x(1-x)=yz \quad (6) \\ \alpha\delta=\beta\gamma \quad (7) \end{gathered}$$

for given $\alpha, \beta, \gamma, \delta$ satisfying (7). Hence 6 unknowns a, b, c, x, y, z and 6 equations.

To make the proofs easier, we mention some consequences of these seven equations.

Multiplying equation (2) by x, and using (1) and (6), gives $\alpha y = \beta x$. Similarly, multiplying (3) by a, and using (1) and (5), we get $\alpha c = \gamma a$.

First we deal with matrices of form $A = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$.

Proposition 4. *A has property 2I in any of the cases below.*
(i) α divides β or β divides α;
(ii) α or β is a unit.

Proof. (i) Indeed $\begin{bmatrix} a & ab \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a+b-1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1-b & b-b^2 \\ 1 & b \end{bmatrix}$ and $\begin{bmatrix} ab & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ b & 1 \end{bmatrix}$.

(ii) If (say) $\alpha \in U(R)$, we write $\beta = \alpha(\alpha^{-1}\beta)$ and apply (i). □

Remark. Already in [1], the following decomposition was given:
$\begin{bmatrix} a & ab \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$. It was only in [2], that the problem of *minimizing* the number of idempotents occurred. For this matrix, a two idempotents decomposition was *left to the reader* ! In the previous proof we disclosed it. These decompositions hold over *not* necessarily commutative rings. For the sake of completeness, recall from [2], that over a *not* necessarily commutative ring, $\begin{bmatrix} ab & b \\ 0 & 0 \end{bmatrix}$ has *not* the 2I property.

Our first main result is the following

Theorem 5. *Let R be a commutative domain. The matrix* $A = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$ *has property 2I if and only if* $\alpha = 0$ *and/or* $\beta = 0$*, or else, there exists* $b \in R$ *such that* $\beta - b\alpha \neq 0$ *divides* $\alpha(1-\alpha)$*. Equivalently, the quadratic polynomial equation*

$$\alpha bz - \beta z + \alpha(1-\alpha) = 0$$

in the unknowns b, z has solutions in R. In particular, this holds if $\alpha \neq 0$ *divides* β *or else,* $\beta \neq 0$ *divides* α *or* $\alpha(1-\alpha)$.

Proof. The system (SI) now becomes

$$\begin{array}{c} ax + bz = \alpha \quad (1) \\ ay + b(1-x) = \beta \quad (2) \\ cx + (1-a)z = 0 \quad (3) \\ cy + (1-a)(1-x) = 0 \quad (4) \\ a(1-a) = bc \quad (5) \\ x(1-x) = yz \quad (6) \end{array}.$$

Since the cases $\alpha = 0$ or $\beta = 0$ were already covered by Lemma 2, we assume $\alpha \neq 0$. The extra equalities deduced in starting this section now yield $\alpha y = \beta x$ and $\alpha c = 0$ whence $c = 0$. Therefore, from (5), $a \in \{0, 1\}$ and we distinguish two cases.

Case 1: $a = 0$. In this case $E = \begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix}$, and the system reduces to $bz = \alpha$, $b(1-x) = \beta$, $z = 0$, $x = 1$ and so $\beta = 0$.

Case 2: $a = 1$. In this case, $E = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$, and the system reduces to $x + bz = \alpha$, $y + b(1-x) = \beta$, $x(1-x) = yz$.

Replacing $x = \alpha - bz$, $y = \beta - b(1-x) = \beta - b(1-\alpha) - b^2 z$ in $x(1-x) = yz$ we obtain $\alpha(1-\alpha) = (\beta - b\alpha)z$. Hence $z = \dfrac{\alpha(1-\alpha)}{\beta - b\alpha}$ exists if and only if $\beta - b\alpha \neq 0$ divides $\alpha(1-\alpha)$ for some b, divisibility which we can write as the quadratic polynomial equation in the unknowns b, z from the statement. Then $x = \alpha - bz = \dfrac{\alpha(\beta - b)}{\beta - b\alpha}$ and $y = \beta - b(1-x) = \dfrac{\beta(\beta - b)}{\beta - b\alpha}$ and so $E = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} \dfrac{\alpha(\beta-b)}{\beta-b\alpha} & \dfrac{\beta(\beta-b)}{\beta-b\alpha} \\ \dfrac{\alpha(1-\alpha)}{\beta-b\alpha} & \dfrac{\beta(1-\alpha)}{\beta-b\alpha} \end{bmatrix}$. It is easy to check $EF = A$.

The cases $\alpha \neq 0$ divides β and $\beta \neq 0$ divides α are covered by Proposition 4. It just remains to consider the case $\beta \neq 0$ divides $\alpha(1-\alpha)$. For this $b = 0$, $x = \alpha$, $y = \beta$ and $z = \dfrac{\alpha(1-\alpha)}{\beta}$. □

Remarks. 1) Over $\mathbb{Z}$, these quadratic Diophantine equations can be solved using [3] or [13].

2) **The example** from [2]: $\alpha = 14$, $\beta = 8$. Since 14, 8 do not divide each other, we check if there exists $b \in \mathbb{Z}$ such that $8 - 14b$ ($\neq 0$) divides $14(1-14) = -182$. Equivalently, $4 - 7b \neq 0$ divides $91 = 13 \cdot 7$, which clearly fails. So indeed, $\begin{bmatrix} 14 & 8 \\ 0 & 0 \end{bmatrix}$ has *not* the 2I property.

Alternatively, the quadratic Diophantine equation $14bz - 8z - 182 = 0$ has no integer solutions.

However, dividing the entries by 2, $\begin{bmatrix} 7 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -7 & -4 \\ 14 & 8 \end{bmatrix}$, has property 2I (for $b = 1$ in the decomposition given in the previous proof).

3) The statement, *if* α, β *are coprime then* $\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$ *has property 2I*, fails. An example is $\begin{bmatrix} 30 & 77 \\ 0 & 0 \end{bmatrix}$. Indeed, the quadratic Diophantine corresponding equation $30bz - 77z - 30 \cdot 29 = 0$ has no integer solutions. The astute reader will notice that $30 = 2 \cdot 3 \cdot 5$, $77 = 7 \cdot 11$. A generalization could be in order: let $p_1 < p_2 < p_3 < p_4 < p_5$ prime numbers. The equation

$$p_1 p_2 p_3 xy - p_4 p_5 y - p_1 p_2 p_3 (p_1 p_2 p_3 - 1) = 0$$

has no integer solutions. This holds for instance for $\{3, 5, 7, 11, 13\}$ or $\{2, 3, 7, 11, 13\}$ or $\{5, 7, 11, 13, 17\}$, but fails for $\{2, 3, 5, 31, 61\}$.

4) A converse, i.e., *if* α, β *are nonzero,* $\gcd(\alpha; \beta) \neq 1$ *and none divides the other, then* $\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$ *has not property 2I*, also fails.

Example: $\begin{bmatrix} 12 & 8 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -21 & -14 \\ 33 & 22 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -3 & -2 \end{bmatrix}$.

5) The reader can see that *the previous characterization is not left-right symmetric*. Indeed, while $\begin{bmatrix} 8 & 14 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 8 & 14 \\ -4 & -7 \end{bmatrix}$
$= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -48 & -84 \\ 28 & 49 \end{bmatrix}$ has the 2I property, $\begin{bmatrix} 14 & 8 \\ 0 & 0 \end{bmatrix}$ does *not* have it.

Recall that the discriminant of a quadratic polynomial equation in two indeterminates, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is $\Delta = B^2 - 4AC$.

Our second main result is

Theorem 6. *Let* $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ *be a singular matrix with nonzero entries over a commutative domain R, i.e., $\alpha\delta = \beta\gamma$. The matrix A has property 2I if and only if the equation*

$$(\alpha + \delta)ax - \alpha(a + x) = \alpha(\delta - 1)$$

in unknowns a, x has at least one solution (a, x) for which α divides βx, β divides $\alpha(1 - x)$, γ divides $\alpha(1 - a)$ and α divides γa.

In particular, this holds in any of the following cases:
(i) β divides $\alpha(1 - \alpha)$ and α divides γ,
(ii) α divides β and γ divides $\alpha(1 - \alpha)$,
(iii) α divides $\beta(1 - \gamma)$ and γ divides α,
(iv) β divides α and α divides $\gamma(1 - \delta)$.

Proof. We come back to the system (SI) with now all $\alpha, \beta, \gamma, \delta \neq 0$. First notice two more consequences of the equations (1)-(7): multiplying (1) by $1 - x$ and using (6) and (2) we get $\beta z = \alpha(1 - x)$, and, multiplying (1) by $1 - a$ and using (5) and (3) we obtain $\gamma b = \alpha(1 - a)$.

Multiplying (1) by $\beta\gamma$ and using the previous relations, we obtain $\beta\gamma ax + \alpha^2(1 - a)(1 - x) = \alpha\beta\gamma$. Using $\beta\gamma = \alpha\delta$, dividing by α and calculating, we finally obtain

$$(*) \quad (\alpha + \delta)ax - \alpha(a + x) = \alpha(\delta - 1)$$

which is a quadratic polynomial equation in the unknowns a, x over the commutative domain R.

Since its discriminant $\Delta = (\alpha + \delta)^2$ is a square, this is the so-called *simple hyperbolic* case, which, if $\alpha + \delta \neq 0$, reduces to

$$(**) \quad [(\alpha + \delta)a - \alpha][(\alpha + \delta)x - \alpha] = \alpha\delta(\alpha + \delta - 1).$$

It is easy to see that choosing $(\alpha+\delta)a-\alpha = -\alpha$, $(\alpha+\delta)x-\alpha = -\delta(\alpha+\delta-1)$ gives the solution $(a, x) = (0, 1-\delta)$, and choosing $(\alpha + \delta)a - \alpha = \delta$, $(\alpha + \delta)x - \alpha = \alpha(\alpha + \delta - 1)$ gives the solutions $(a, x) = (1, \alpha)$. Symmetrically we also get $(a, x) = (1 - \delta, 0)$ and $(a, x) = (\alpha, 1)$.

To simplify the writing, we formally use fractions, for the other four unknowns y, z, b, c, expressed with respect to a and x: $y = \frac{\beta x}{\alpha}$, $z = \frac{\alpha(1 - x)}{\beta}$, $b = \frac{\alpha(1 - a)}{\gamma}$ and $c = \frac{\gamma a}{\alpha}$. Replacing $(1, \alpha)$ gives the divisibilities in (i). Replacing $(\alpha, 1)$, $(0, 1 - \delta)$ and $(1 - \delta, 0)$ give the divisibilities in (ii), (iii) and (iv), respectively. □

Remarks. 1) Over $\mathbb{Z}$ the equation (**) is easily solved by decomposing $\alpha\delta(\alpha + \delta - 1)$ into factors and so gives finitely many solutions. As mentioned in the previous proof, among these we always have the solutions $(0, 1-\delta)$, $(1 - \delta, 0)$, $(1, \alpha)$ and $(\alpha, 1)$, which we call, *exceptional* solutions. Often, the (*) (or (**)) equation has only these (four) exceptional solutions. But not always: for $\alpha = 20$ and $\delta = 45$, we have the exceptional solutions $(1, 20)$, $(20, 1)$, $(-44, 0)$ and $(0, -44)$ but also $(4, 4)$. As expected, the number of solutions increases whenever $\alpha\delta(\alpha + \delta - 1)$ has many two factors decompositions.

2) A special case it worth mentioning: $\alpha+\delta = 0$, that is, singular matrices of form $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ (so $\alpha^2+\beta\gamma = 0$). By Cayley-Hamilton's theorem, these are precisely the nilpotents, already discussed in Corollary 3.

Examples. 1) $A = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$. The (*) equation is $5ax - 2(a + x) - 4 = 0$ which has only the exceptional solutions $(0, -2)$, $(-2, 0)$, $(1, 2)$ and $(2, 1)$.

For $(a, x) = (-2, 0)$ we get the decomposition $A = \begin{bmatrix} -2 & 2 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. Indeed, as in (iv), $\beta = 2$ divides $\alpha = 2$ and $\alpha = 2$ divides $\gamma(1-\delta) = -6$.

2) $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. The (*) equation is $7ax - (a + x) - 5 = 0$ which has only the exceptional solutions $(0, -5)$, $(-5, 0)$, and $(1, 1)$. For $(a, x) = (1, 1)$ we get the decomposition $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Indeed, as in (i) (or (ii)), $\beta = 2$ divides $\alpha(1-\alpha) = 0$ and $\alpha = 1$ divides $\gamma = 3$ (resp. $\alpha = 1$ divides $\beta = 2$ and $\gamma = 3$ divides $\alpha(1-\alpha) = 0$).

3) $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$. The (*) equation is $8ax - 2(a + x) - 10 = 0$ which has only the exceptional solutions $(0, -5)$, $(-5, 0)$, $(1, 2)$ and $(2, 1)$. None of the necessary (and sufficient) divisibilities, listed in the statement of the previous theorem, holds. More precisely, the corresponding decompositions over $\mathbb{Q}$ are:

$\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -\frac{2}{3} & -1 \end{bmatrix}$ for $(a, x) = (1, 2)$ as in (i),

$\begin{bmatrix} 2 & -\frac{1}{2} \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 0 \end{bmatrix}$ for $(a, x) = (2, 1)$ as in (ii),

$\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -5 & -\frac{15}{2} \\ 4 & 6 \end{bmatrix}$ for $(a, x) = (0, -5)$ as in (ii), and

$\begin{bmatrix} -5 & 3 \\ -10 & 6 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \frac{2}{3} & 1 \end{bmatrix}$ for $(a, x) = (1, 2)$ as in (i). Hence, this matrix has *not* property 2I.

4 Matrices with 2N

We start with

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = ST = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}\begin{bmatrix} x & y \\ z & -x \end{bmatrix} =$$

$= \begin{bmatrix} ax + bz & ay - bx \\ cx - az & cy + ax \end{bmatrix}$ with $\alpha\delta = \beta\gamma$, $a^2 + bc = 0$ and $x^2 + yz = 0$. With respect to the 2N property, a system denoted (SN) (again) with 6 unknowns a, b, c, x, y, z and 6 equations, analogous with (SI) (see Section 3) has to be solved.

Since we are searching for two nilpotent factors decompositions of singular matrices, as already mentioned in Section 2, the same cases must be addressed. Since the computations are analogous we skip the proofs of our two next results.

Theorem 7. *Let R be a commutative domain. The matrix* $A = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$ *has property 2N if and only if* $\beta = 0$ *or* $\alpha, \beta \neq 0$ *and* α *divides* β^2.

Theorem 8. *Let* $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ *be a singular matrix with nonzero entries over a commutative domain R, i.e.,* $\alpha\delta = \beta\gamma$*. The matrix A has property 2N if and only if* $\alpha + \delta$ *divides* $\alpha\delta$ *and the equation*

$$(\alpha + \delta)ax = \alpha\delta$$

in the unknowns a, x *has at least one solution* (a, x) *for which* α *divides* βx*,* β *divides* αx*,* γ *divides* αa *and* α *divides* γa*.*

Corollary 9. *Nonzero* 2×2 *nilpotent matrices over any commutative domain do not have property 2N.*

Proof. Notice that if $\alpha + \delta = 0$ (with nonzero α, δ) then the equation above has no solutions. Consequently, nilpotents (with only nonzero entries) have *not* property 2N. Since nilpotents with three zeros or with zero second row do not have property 2N, the statement follows. □

Remark. Over $\mathbb{Z}$, if (the nonzero) α, δ have the same sign, then the only matrices with 2N property arise for $(\alpha, \delta) \in \{(\pm 1, \pm 1), (\pm 2, \pm 2)\}$.

Examples. 1) $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

2) $A = \begin{bmatrix} 3 & 3 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$.

3) $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$. Here $2 + 2$ divides $1 \cdot 4$ and the equation $4ax = 4$ has only two solutions (a, x): $(1, 1)$ and $(-1, -1)$. None verifies the required divisibilities, so A has *not* property 2N.

4) $A = \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix}$. Here $3 - 2$ divides $3 \cdot (-2)$ and the equation $ax = -6$ has several solutions (a, x): $(\pm 1, \mp 6)$, $(\pm 2, \mp 3)$ and symmetric. Only $(a, x) = (\pm 3, \mp 2)$ verify the required divisibilities. For instance, $A = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$.

References on decompositions of singular matrices into products of nilpotent matrices *over fields*, include [16], [14] and [15].

5 Comments and an open question

We could wonder what can be said about 3×3 matrices (over commutative domains) with respect to the properties 2I or 2N. Cayley-Hamilton's theorem,

$$A^3 - \mathrm{Tr}(A)A^2 + \frac{1}{2}(\mathrm{Tr}^2(A) - \mathrm{Tr}(A^2))A - \det(A)I_3 = 0_3,$$

still gives the form of the nonzero 3×3 nilpotent matrices, i.e., $\mathrm{Tr}(A) = \frac{1}{2}(\mathrm{Tr}^2(A) - \mathrm{Tr}(A^2)) = \det(A) = 0$, but requires some additional conditions in order to obtain the nontrivial 3×3 idempotents: a 3×3 matrix E over an ID GCD domain R is nontrivial idempotent if and only if $\det(E) = 0$, $\mathrm{rank}(E) = \mathrm{Tr}(E) = 1 + \frac{1}{2}(\mathrm{Tr}^2(E) - \mathrm{Tr}(E^2))$ and $\mathrm{rank}(E) + \mathrm{rank}(I_3 - E) = 3$ (for a proof see [5]).

Here R is an *ID* ring if every idempotent matrix over R is similar to a diagonal one. Examples of ID rings include: division rings, local rings, projective-free rings, PID's, elementary divisor rings, unit-regular rings and serial rings. This way, the problems become far more complicated.

Anyway, the property mentioned in the previous section: 2×2 *nilpotents over any commutative domain do not have property 2N*, fails for 3×3 matrices. More generally, in any ring R, for every nilpotent element t

of index $n \geq 3$, the nilpotents $t^2, t^3, \ldots, t^{n-1}$ trivially have the property 2N. Therefore, it is natural to ask the following

Question. What are the (matrix) rings for which the nonzero zero-square elements (resp. matrices) do not have the property 2N?

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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