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SPLIT-EXTENSIONS OF EXCHANGE RINGS: A DIRECT PROOF

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A direct ring-theoretic proof for "if R is a ring, e is an idempotent and eRe and $(1-e)R(1-e)$ are both exchange rings then R is also an exchange ring" (left open for the last 31 years) is given.

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0. Introduction

The exchange property was first (introduced and) studied for modules by Crawley and Jónsson in [1]: for a given cardinal m, a right R-module M_R has the *m-exchange property* if whenever $A = M \oplus N = \bigoplus_{i\in I} A_i$ with $|I| \leq m$, there are submodules $A'_i \leq A_i$ with $A = M \oplus (\bigoplus_{i\in I} A'_i)$. If M has m-exchange for all cardinals m, we say M has the *full exchange* property. If the same holds just for finite cardinals, we say M has the *finite exchange* property. Further on, Warfield, Jr. in [5], considered this property for rings (with identity): a ring R such that R_R has finite exchange is called (right) *exchange* ring, and this turned out to be a left-right symmetric condition.

In his 1977 seminal paper "Lifting idempotents and exchange rings", among other things (starting point of clean rings theory, characterization of exchange rings by lifting idempotents modulo one-sided ideals, etc), Nicholson calls a ring (right) *suitable* if for each equation $a + b = 1$ there are (orthogonal) idempotents $e \in aR$ and $f \in bR$ such that $e + f = 1$. Suitable rings and exchange rings coincide.

In the sequel, by exchange ring we mean right exchange (suitable) ring and assume/check this last (equivalent) condition.

From this paper we quote: "It would be of interest to see a direct ring-theoretic proof of the fact that *if R is a ring, $e \in R$ is an idempotent and eRe and $(1-e)R(1-e)$ are both exchange rings then R is also an exchange ring*". No such direct proof seems to have appeared the last 31 years. This paper closes this gap.

1. The Claim

For convenience write $\overline{r} = 1 - r$ for each $r \in R$. For an idempotent $e \in R$, we use the Pierce decomposition of the ring R:

$$R = \begin{bmatrix} eRe & eR\overline{e} \\ \overline{e}Re & \overline{e}R\overline{e} \end{bmatrix}.$$

Let $A = \left[\begin{smallmatrix} a & x \\ y & \alpha \end{smallmatrix}\right] \in R$. Since $a \in eRe$ (ring with identity e) is exchange, there are $b, c \in eRe$ such that $bab = b$ and $(e-a)c = e-ab$. The element c can be chosen (see Sec. 2(b)), such that $c = c(e-a)c$. Consider $\alpha_1 = \alpha - y(b-c)x \in \overline{e}R\overline{e}$. By exchange property, there are $\beta, \gamma \in \overline{e}R\overline{e}$ such that $\beta\alpha_1\beta = \beta$ and $(\overline{e} - \alpha_1)\gamma = \overline{e} - \alpha_1\beta$. Once again (by Sec. 2(b)), $\gamma = \gamma(\overline{e} - \alpha_1)\gamma$ can be assumed.

Taking

$$C = \begin{bmatrix} c + cx(\beta-\gamma)y(b-c) & -cx(\beta-\gamma) \\ -\gamma y(b-c) & \gamma \end{bmatrix}$$

and

$$B = \begin{bmatrix} b + bx(\beta-\gamma)y(b-c) & -bx(\beta-\gamma) \\ -\beta y(b-c) & \beta \end{bmatrix},$$

one verifies the equalities

$$BAB = B \quad \text{and} \quad I_2 - AB = (I_2 - A)C$$

where

$$I_2 = \begin{bmatrix} e & 0 \\ 0 & \overline{e} \end{bmatrix}.$$

Therefore, $E = AB \in AR$ is an idempotent and $I_2 - E \in (I_2 - A)R$, as desired.

2. The Verification

Since (despite the easy claim above), the verification of these two matrix equalities is far from obvious, in this section we supply the necessary details.

We first recall some well-known relations which hold in any exchange ring.

(a) If $e \in aR$ is an idempotent, an element $b \in R$ can be chosen such that $e = ab$ and $bab = b$.

(b) If $1 - e \in (1-a)R$ is idempotent, an element $c \in R$ can be chosen such that $1 - e = (1-a)c$ and $c(1-a)c = c$.

(c) $cab = 0$ [using $c(1-a)c = c$ and left multiplication by c in $1 - ab = (1-a)c$].

Hence, replacing R by the corner ring eRe, these relations hold for the elements a, b, and c in the above claim (written $e - ab = (e - a)c$ from now on denoted (a')).

Similar properties hold for the Greek letters in the above claim (now replacing R by $\overline{e}R\overline{e}$).

In order to simplify the writing, denote $s = x(\beta - \gamma)$ and $t = y(b - c)$. Thus $B = \begin{bmatrix} b + bst & -bs \\ -\beta t & \beta \end{bmatrix}$ and $C = \begin{bmatrix} c + cst & -cs \\ -\gamma t & \gamma \end{bmatrix}$.

Notice the following relations:

(d) $cab = 0$ implies $tab = yb$.
(e) Using $\alpha_1 = \alpha - tx$ in $(\overline{e} - \alpha_1)\gamma = \overline{e} - \alpha_1\beta$, gives $(\overline{e} - \alpha)\gamma = \overline{e} - \alpha\beta + ts$.
(f) Using it in $\beta\alpha_1\beta = \beta$ gives $\beta\alpha\beta = \beta + \beta tx\beta$.
(g) $\gamma\alpha_1\beta = 0$ and finally $\gamma\alpha\beta = \gamma tx\beta$.

2.1. *The equality* $BAB = B$

11-entry: $b(e + st)ab(e + st) - b(e + st)x\beta t - bsyb(e + st) + bs\alpha\beta t = b(e + st)$, follows from $bab = b$ (a), $tab = yb$ (d) and $s = x(\beta - \gamma)$.

12-entry: $-b(e+st)abs + b(e+st)x\beta + bsybs - bs\alpha\beta = -bs$, follows from $bab = b$ (a), $tab = yb$ (d), $\gamma\alpha\beta = \gamma tx\beta$ (g) and $\beta\alpha\beta = \beta + \beta tx\beta$ (f).

21-entry: $-\beta tab(e+st) + \beta tx\beta t + \beta yb(e+st) - \beta\alpha\beta t = -\beta t$ follows from $tab = yb$ (d) and $\beta\alpha\beta = \beta + \beta tx\beta$ (f).

22-entry: $\beta tabs - \beta tx\beta - \beta ybs\beta\alpha\beta = \beta$ follows from $tab = yb$ (d) and $\beta\alpha\beta = \beta + \beta tx\beta$ (f).

2.2. *The equality* $I_2 - AB = (I_2 - A)C$

11-entry: $e - ab(e+st) + x\beta t = (e-a)c(e+st) + x\gamma t$ follows from $(e-a)c = e - ab$ (a') and $s = x(\beta - \gamma)$.

12-entry: $abs - x\beta = -(e - a)cs - x\gamma$ follows from $(e - a)c = e - ab$ (a') and $s = x(\beta - \gamma)$.

21-entry: $-yb(e + st) + \alpha\beta t = -yc(e + st) - (\overline{e} - \alpha)\gamma t$ follows from $(\overline{e} - \alpha)\gamma = \overline{e} - \alpha\beta + ts$ (e) and $t = y(b - c)$.

22-entry: $ycs + (\overline{e} - \alpha)\gamma = \overline{e} + ybs - \alpha\beta$ follows from $(\overline{e} - \alpha)\gamma = \overline{e} - \alpha\beta + ts$ (e) and $t = y(b - c)$.

3. Final Remarks

(1) If eRe and $\overline{e}R\overline{e}$ are both clean, then the elements considered above have (understandably, since clean rings are exchange) stronger properties: $b - c = u_1$ is a unit in eRe and so is $\beta - \gamma = v_1$ (in $\overline{e}R\overline{e}$). If u is the inverse of u_1, v is the inverse of v_1 and $\varepsilon = -cu$, $g = -\gamma v$ then $A = \begin{bmatrix} a & x \\ y & \alpha \end{bmatrix}$ decomposes into a sum of an idempotent and a unit: $E + U = \begin{bmatrix} \varepsilon & 0 \\ 0 & g \end{bmatrix} + \begin{bmatrix} u & x \\ y & v + y(b - c)x \end{bmatrix}$. As such, it is also clean.

In this way we recover the proof of Lemma 1 in [2].

(2) Actually, in the above mentioned paper, Nicholson also asks for a direct ring-theoretic proof for "the (finite) exchange property is a left-right symmetric condition". It took 20 years until Nicholson himself gave this direct proof (see [4]).

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