A COUNTEREXAMPLE ?

Let R be a commutative ring and let $T \in \mathbb{M}_3(R)$ with $\mathrm{Tr}(T) = 0$. To avoid too many indexes and emphasize the diagonal elements (i.e. the zero trace) we write $T = \begin{bmatrix} x & a & c \\ b & y & e \\ d & f & -x-y \end{bmatrix}$. We consider the following two conditions

(A) $T^2 = 0_3$, and

(B) all 2×2 minors of T equal zero, and show that **(B)**$\Longrightarrow$**(A)** but **(A)**$\Longrightarrow$**(B)** only if 2 is not a zero divisor (i.e., is cancellable).

The condition **(A)**, i.e., $T^2 = 0_3$ is *equivalent* to the following ($9 = 3 \times 3$) equalities

$$\begin{array}{c} x^2 + ab + cd = 0 \quad (1) \\ a(x+y) + cf = 0 \quad (2) \\ ae = cy \quad (3) \\ b(x+y) + de = 0 \quad (4) \\ y^2 + ab + ef = 0 \quad (5) \\ bc = ex \quad (6) \\ bf = dy \quad (7) \\ ad = fx \quad (8) \\ (x+y)^2 + cd + ef = 0 \quad (9). \end{array}$$

Denote by T_{ab}^{cd} the 2×2 minor on the rows a and b and on the columns c and d. [The well-known properties of determinants yield $T_{ba}^{cd} = T_{ab}^{dc} = -T_{ab}^{cd}$].

The two terms equalities are *equivalent* to the vanishing of four 2×2 minors. Namely,

(3) $\equiv (T_{12}^{23} = 0)$, (6) $\equiv (T_{12}^{13} = 0)$, (7) $\equiv (T_{23}^{12} = 0)$, (8) $\equiv (T_{13}^{12} = 0)$.

Further, two other equalities are *equivalent* to the vanishing of another two minors. Namely,

(2) $(a(x+y) + cf = 0) \equiv (T_{13}^{23} = 0)$,

and (4) $(b(x+y) + de = 0) \equiv (T_{23}^{13} = 0)$.

Therefore this covers (if and only if) the six **not** diagonal 2×2 minors.

What remains is the vanishing of the three 2×2 *diagonal* minors, i.e.,

$T_{12}^{12} = 0 : xy = ab$, $T_{13}^{13} = 0 : x(-x-y) = cd$ and $T_{23}^{23} = 0 : y(-x-y) = ef$.

No condition needed for (B) $\Longrightarrow$ (A):

$xy = ab$ ($T_{12}^{12} = 0$) and $cd = -x(x+y)$ ($T_{13}^{13} = 0$) imply $x^2 + ab + cd = 0$ (1),

$xy = ab$ ($T_{12}^{12} = 0$) and $ef = -y(x+y)$ ($T_{23}^{23} = 0$) imply $y^2 + ab + ef = 0$ (5), and

$cd = -x(x+y)$ ($T_{13}^{13} = 0$), $ef = -y(x+y)$ ($T_{23}^{23} = 0$) imply $(x+y)^2 + cd + ef = 0$ (9).

Seems that a condition is needed for (A) $\Longrightarrow$ (B).

By hypothesis all equalities (1) - (9) hold. In the 8 letters used, each equality has degree 2. Multiplication of any relation by any letter will increase the degree and the vanishing of the diagonal 2×2 minors cannot be deduced, because cancellation is not possible (unless we assume some non zero divisors; e. g., see the lemma below, (ii)).

We focus on $T^{12}_{12}=0$.

The term ab appears only in (1) and (5), while the term xy appears only in (9).

From $x^2+ab+cd=0$ (1), $y^2+ab+ef=0$ (5) and $(x+y)^2+cd+ef=0$ (9) we get $2xy=2ab$. This implies $xy=ab$ ($T^{12}_{12}=0$) iff 2 is not a zero divisor.

Finally using $x^2+ab+cd=0$ (1), $y^2+ab+ef=0$ (5) and $xy=ab$ ($T^{12}_{12}=0$), we get the last two zero 2×2 diagonal minors: $x(x+y)+cd=0$ ($T^{13}_{13}=0$) and $y(x+y)+ef=0$ ($T^{23}_{23}=0$).

Remark. While in the proof above the hypothesis "2 *is not a zero divisor*" is essential for the vanishing of the three diagonal 2×2 minors (actually for getting $ab=xy$ from $2ab=2xy$), we were not able to find a *square-zero* 3×3 matrix with *zero trace* over a ring where 2 is a zero divisor, *which has a nonzero* 2×2 *minor*. In searching for such an example, the following observations gathered in the following lemma may help.

Lemma 0.1. *(i) Suppose* $T^2=0_3$. *If any diagonal* 2×2 *minor is zero, so are the other two diagonal* 2×2 *minors.*

(ii) Suppose $T^2=0_3$. *If any entry of* T *is not a zero divisor, then all* 2×2 *minors are zero.*

Proof. As noticed in the previous proof, if $T^2=0_3$ then all (the six) not diagonal minors are zero.

(i) In the previous proof we already saw that $xy=ab$ implies $x(x+y)+cd=0$ and $y(x+y)+ef=0$.

If $x(x+y)+cd=0$, combining with $x^2+ab+cd=0$ we get $xy=ab$ and so the third diagonal minor vanishes.

If $y(x+y)+ef=0$, combining with $y^2+ab+ef=0$ we get $xy=ab$ and so the third diagonal minor vanishes.

(ii) As noticed in the proof of the previous theorem, our concern are *the diagonal* 2×2 minors. The proof can be done separately for each entry.

If x is cancellable, we multiply $x^2=-ab-cd$ by y and so $x^2y\overset{ae=cy}{=}-aby-ade=-a(by+de)\overset{b(x+y)+de=0}{=}-a(-bx)=abx$. By cancellation we get $xy=ab$ and so the other two diagonal 2×2 minors, using (i).

If a is cancellable, we multiply $a(x+y)+cf=0$ by x and using $ad=fx$ we obtain $x(x+y)+cd=0$ and the other two by (i).

If c is cancellable, we multiply $bc=ex$ by a and using $ae=cy$ we obtain $ab=xy$ and the other two.

If b is cancellable, we multiply $b(x+y)+de=0$ by y and using $bf=dy$ we obtain $y(x+y)+ef=0$ and the other two.

If y is cancellable, we multiply $y^2=-ab-ef$ by x and using $fx=ad$ and $bx+de=-by$ we obtain $aby=xy^2$. By cancellation we get $xy=ab$ and the other two.

If e is cancellable, we multiply $bc=ex$ by y and using $ae=cy$ we obtain $ab=xy$ and the other two.

If d is cancellable, we multiply $bf = dy$ by $x + y$ and using $b(x + y) + de = 0$ we obtain $y(x + y) + ef = 0$ and the other two.

If f is cancellable, we multiply $ad = fx$ by y and using $bf = dy$ we obtain $ab = xy$ and the other two.

If $x+y$ is cancellable, we multiply $(x+y)^2 = -cd-ef$ by x and using $bc = ex$ and $bf = dy$ we obtain $x(x+y)^2 = -cd(x+y)$. By cancellation we get $x(x+y)+cd = 0$ and the other two. □

In conclusion, we are searching for *a commutative ring with* 2 *being zero divisor, and a* 3×3 *matrix* T*, all whose entries are zero divisors, such that* $T^2 = 0_3$ *but all the diagonal* 2×2 *minors are nonzero.*

Even more precisely, with the above notations, we need

$2xy = 2ab$ but $xy \neq ab$,

$2x(x + y) + 2cd = 0$ but $x(x + y) + cd \neq 0$ and

$2y(x + y) + 2ef = 0$ but $y(x + y) + ef \neq 0$.