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Strongly inert subgroups of Abelian groups

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Abstract – Mixing in a natural way the notions of fully inert (see [6]) and strongly invariant (see [4]) subgroups of Abelian groups, we introduce the strongly inert subgroups which we determine for several classes of Abelian groups.

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1. Introduction

Dikranian, Giordano Bruno, Goldsmith, Salce, Virili and Zanardo defined and studied fully inert subgroups of Abelian groups in [6], [7], [8], [10] and [11]. Strongly invariant subgroups of Abelian groups were defined and studied by the second named author in [4]. A natural mixture of these two notions (suggested by P. Danchev) gives a new notion which we call strongly inert subgroup.

The relations among these notions are resumed in the following diagram

$$\begin{array}{ccccc} & & \text{strongly} - \text{invariant} & & \\ & \swarrow & & \searrow & \\ \text{strongly} - \text{inert} & & & & \text{fully} - \text{invariant} \\ & \searrow & & \swarrow & \\ & & \text{fully} - \text{inert} & & \end{array}$$

After some prerequisites (Section 2 contains the definitions, the above relations between all the involved notions and the examples proving that these implications

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cannot be reversed) and properties of strongly inert subgroups in relation to those of strongly invariant ones (Section 3), having, to some extent, as models the above mentioned investigations, in Section 4 of this paper we determine the strongly inert subgroups of some classes of Abelian groups, namely, of divisible groups, free groups and torsion groups. As main result, we prove that a subgroup of a torsion group is strongly inert if and only if it is commensurable with a strongly invariant subgroup.

In the sequel, the word "group" means Abelian group. For standard notations, terminology and results we refer to the authoritative book of Laszlo Fuchs (see [9]). For a positive integer n, the cyclic group with n elements is denoted by $\mathbf{Z}(n)$ and, for a group G we denote by $G[n]$ the set of elements of G whose order is a divisor of n and by $S(G)$, the socle of G, that is the set of elements of G of square-free order. For a prime p, and a group G, G_p denotes the p-component of G. We will use the widely accepted shorthand "iff" for "if and only if" in the text.

2. Prerequisites

For a group G and a subgroup N we denote by ϕ, endomorphisms of G and by $f : N \longrightarrow G$, homomorphisms. First recall the following

Definition 1. A subgroup H of G is called *ϕ-invariant* if $\phi(H) \subseteq H$ and *fully invariant* if it is ϕ-invariant for every $\phi \in \mathrm{End}(G)$.

Definition 2. A subgroup H is called *ϕ-inert* (see [7]) if $\phi(H) \cap H$ has finite index in $\phi(H)$, or equivalently, if the factor group $(\phi(H) + H)/H$ is finite, and *fully inert* if it is ϕ-inert for every endomorphism of G.

Then clearly if $|H|$ or $|G : H|$ is finite then H is fully inert, ϕ-invariant subgroups are ϕ-inert and so fully invariant subgroups are fully inert.

Definition 3. A subgroup N of G is called *f-invariant* (see [4]) if $f(N) \subseteq N$ and *strongly invariant* if it is f-invariant for every $f \in \mathrm{Hom}(N, G)$.

Definition 4. A subgroup N is called *f-inert* if $f(N) \cap N$ has finite index in $f(N)$, or equivalently, if the factor group $(f(N) + N)/N$ is finite, and *strongly inert* if it is f-inert for every homomorphism $f : N \longrightarrow G$.

Then again finite subgroups and subgroups of finite index are strongly inert, f-invariant subgroups are f-inert and so *strongly invariant subgroups are strongly inert.*

Therefore, in a given group, for subgroups, we have the inclusions

$\{\text{strongly invariant}\} \overset{1}{\subset} \{\text{strongly inert}\} \overset{2}{\subset} \{\text{fully inert}\}$, and

$\{\text{strongly invariant}\} \overset{3}{\subset} \{\text{fully invariant}\} \overset{4}{\subset} \{\text{fully inert}\}$.

None of these inclusions is reversible (examples are given below).

Determining fully inert subgroups of Abelian groups was (as References show) a difficult task. As expected, the determination of strongly inert subgroups is simpler.

A useful notion in the study of fully/strongly inert subgroups is given by the following

Definition 5. Two subgroups H and K of a group G are called *commensurable* if both $(K+H)/H$ and $(K+H)/K$ are finite (i.e., $H \cap K$ has finite index in both H and K).

Recall that this notion comes back to [2], where (as in [1] and [12]) a special case of ϕ-inert subgroup, for a conjugation automorphism ϕ, is studied for not necessarily commutative groups. A kindred notion called *inertial automorphism* (reversing the rôles of the subgroup and of ϕ), is studied for Abelian groups in [5].

Obviously any two subgroups are commensurable in a finite (or with only finite quotients) group G. If the subgroups are comparable, say $K \leq H$, then these are commensurable iff H/K is finite.

Notice that two rational groups (i.e., subgroups of $\mathbf{Q}$) are commensurable iff these have the same type (are isomorphic).

Examples. a) All subgroups of $\mathbf{Q}$ are fully inert (see [7]). Actually these are also strongly inert. Indeed, let $A \leq \mathbf{Q}$ and let $f : A \longrightarrow \mathbf{Q}$ be a nonzero homomorphism. Since these are rank 1 groups, f must be mono. Therefore $A \cong f(A)$ and so these have the same type. Moreover, they are commensurable and so $A \cap f(A)$ has finite index in $f(A)$. Hence A is strongly inert.

b) Examples for the proper inclusions numbered above.

[**1**] Consider the rational group $\mathbf{Q}_p = \{\frac{m}{n} \in \mathbf{Q} | \gcd(n;p) = 1\}$. Since $\mathbf{Q}_p/p\mathbf{Q}_p \cong \mathbf{Z}(p)$, the cyclic group of order p, this subgroup has finite index and so is strongly inert. It is not strongly invariant according to [4, Lemma 23].

Notice that this example also shows that a finite-index subgroup, which is both strongly inert and fully invariant, need not be strongly invariant. Likewise, finite subgroups which are both fully invariant and strongly inert, need not be strongly invariant (e.g. a finite subgroup of $\mathbf{Z}(2) \oplus \mathbf{Z}(4)$; see [4, Example 2.1]).

[**2**] A subgroup which is not strongly inert, but is fully invariant and so, fully inert too, is given below at (c).

[**3**] The multiplication with $\frac{1}{p}$ shows that $p\mathbf{Q}_p$ - which is fully invariant in $\mathbf{Q}_p$ - is not strongly invariant in $\mathbf{Q}_p$. Being fully invariant, this subgroup is also fully inert.

[**4**] A torsion-free group is fully invariant simple (i.e. has no non-trivial fully invariant subgroups) iff it is divisible. So $\mathbf{Q}$ has no proper fully invariant subgroups but all subgroups of $\mathbf{Q}$ are fully inert.

c) *The properties strongly inert and fully invariant are independent*. As seen above, every proper subgroup of $\mathbf{Q}$ is strongly inert but not fully invariant. Conversely, take a free rank 1 subgroup H of a rank two torsion-free group G which has the endomorphism ring $\mathrm{End}(G) \cong \mathbf{Z}$. All subgroups of G are fully invariant but H is not strongly inert.

3. Comparison with strongly invariant

In this section we revisit the properties **SI1-SI9** of strongly invariant subgroups from [4] (we skip **SI4** which refers to not necessarily commutative groups), from the strongly inert point of view. Most of the examples there, are not suitable because they were given in finite groups. However, many of them can be adapted.

The reason for considering these nine metaproperties is that they are necessary in many routine group-theoretic proofs.

For the sake of completeness, for each property we recall the status for strongly invariant subgroups.

Before starting we just record here a few straightforward but useful results

Proposition 3.1. *(a) In any group G, for any positive n, the subgroup $G[n]$ is strongly inert.*

(b) A subgroup N is strongly inert in G iff for any subgroup H of G, $(H+N)/N$ is finite whenever a subgroup epimorphism $N \longrightarrow H$ exists.

(c) Fully inert direct summands are strongly inert.

(d) Let G be a torsion-free group of finite rank. Then every finitely generated subgroup H of maximal rank is strongly inert.

SI1) *The strongly inert property is not transitive.*

Example. For an infinite set I consider $G = H \oplus L = (\bigoplus_{i\in I} \mathbf{Z}(2^\infty)) \oplus (\bigoplus_{i\in I} \mathbf{Z}(2))$ and take $K = H[2] = S(H)$ the socle (where $\mathbf{Z}(2^\infty)$ is the Prüfer group). Then K is strongly invariant in H (see [4, Proposition 8]) and so also strongly inert in H. As fully invariant direct summand of G, H is strongly invariant in G and so also strongly inert. Finally, K is not strongly inert in G. Indeed, take as homomorphism $f : K \longrightarrow G$, the composition of the isomorphism $K \cong L$ and the injection $i_L : L \longrightarrow G$. Then $(f(K) + K)/K = (L + K)/K \cong L/(L \cap K) = L$ is not finite.

Remarks. 1) This example shows more: if $K \leq H \leq G$, K is strongly invariant in H and H is strongly invariant in G, then K may not be (even) strongly inert in G.

2) In all the references we have mentioned, an example which shows that the property fully inert is not transitive is missing. The previous example cannot be used for this purpose. Indeed, it is easy to check that *if K is fully inert in H and H is fully invariant in G then K is fully inert in G*. Therefore, so far, to give an example which shows that the property fully inert is not transitive, remains an open question.

In the same vein

Proposition 3.2. *Given a group G and subgroups K, H of G, if K is a fully inert direct summand of H with H/K finite and H is strongly inert in G then K is strongly inert in G.*

Proof. For $K \leq H \leq G$ suppose $H = K \oplus L$, for some $L \leq G$, and let $f : K \longrightarrow G$ be any homomorphism. If $\pi_K^H : H \longrightarrow K$ denotes the projection then $\overline{f} = f \circ \pi_K^H : H \longrightarrow G$ and so $\frac{\overline{f}(H)+H}{H}$ is finite, by hypothesis. Since $\overline{f}(K) = f(K)$

and $\frac{\overline{f}(H)+H}{K}$ is finite, $\frac{f(K)+K}{K} \leq \frac{\overline{f}(H)+H}{K}$ is also finite, as required (indeed, we consider the canonical surjective homomorphism $\pi : \frac{\overline{f}(H)+H}{K} \longrightarrow \frac{\overline{f}(H)+H}{H}$; since $\ker \pi = H/K$ is finite, $\frac{\overline{f}(H)+H}{K}$ is finite iff $\frac{\overline{f}(H)+H}{H}$ is finite). □

Observe that above K is a direct summand of H, commensurable with H.
The strongly invariant property is also not transitive.

SI2) *If G is a group and $K \leq H \leq G$ such that K is a strongly inert subgroup in G then K is also strongly inert subgroup in H.*

A straightforward verification shows that *if $K \leq H$, H is a direct summand of a group G and K is fully inert in G, then K is fully inert in H.* Notice that the direct summand property may be weakened requiring $K \in \mathcal{W}(H)$, if $\mathcal{W}(H)$ denotes the class of all subgroups K of H such that all endomorphisms of K can be extended to endomorphisms of H (see [3]). The "strongly" analogue is immediate.

The similar property holds also for strongly invariant subgroups.

SI3) The family of all the strongly inert subgroups of a group is closed under finite sums, but *not* under intersections.

Lemma 3.3. *If H and H' are strongly inert subgroups of a group G, then $H+H'$ is also strongly inert.*

Proof. Take a homomorphism $f : H + H' \longrightarrow G$. Then $[H + H' + f(H + H')]/(H + H')$ is a quotient of the direct sum of finite groups $[(H + f(H))/H] \oplus [(H' + f(H'))/H']$ (here the restrictions of f to both H and H' are also denoted f) and so it is finite too. □

However, intersections of strongly inert subgroups may not be strongly inert.

Example. We again use the example in **SI1**, where H is strongly inert in G. Further, the socle $S(G)$ is also strongly invariant in G and so also strongly inert. Finally, $H \cap S(G) = K$ is not strongly inert in G.

Since K is not strongly invariant this example also covers the missing example in [4] (that is, intersections of strongly invariant subgroups may not be strongly invariant).

Since Lemma **2.3** from [7] was already adapted for strongly inert subgroups (Proposition 3.1, (d)), we can use Example **2.7** in [7] in order to show that sums of infinitely many strongly inert subgroups may not be strongly inert.

The family of all the strongly invariant subgroups is closed under arbitrary sums, but not under intersections.

SI5) Conjecture. *If G is a group and $K \leq H \leq G$ such that K is strongly inert in G and H/K is strongly inert in G/K, then H may not be strongly inert in G.*

Although we suspect that this statement is true, we were not able to find an example.

However the implication holds by replacing K strongly inert in G by K strongly invariant in G:

Lemma 3.4. *If G is a group and $K \leq H \leq G$ such that K is strongly invariant in G and H/K is strongly inert in G/K, then H is strongly inert in G.*

Proof. Start with a homomorphism $f : H \longrightarrow G$. Since K is strongly invariant, f naturally induces a homomorphism $\overline{f} : H/K \longrightarrow G/K$ (i.e., $\overline{f}(h+K) = f(h)+K$). By hypothesis $[(H/K)+\overline{f}(H/K)]/(H/K) = [(H/K)+(f(H)+K)/K]/(H/K) \cong (f(H)+H)/H$ is finite, as required. □

However, the similar property holds for strongly invariant subgroups.

SI6) *If H is strongly inert in a group G, then in any finite direct power of G^n, the corresponding direct power H^n is strongly inert.*

Proof. We give a proof in the $n = 2$ case; the general case is analogous. Any homomorphism $f : H \times H \longrightarrow G \times G$ induces four homomorphisms $f_{ij} = \pi_j \circ f \circ \iota_i : H \longrightarrow G$ where π_j denotes the canonical projection onto the j-th component of $G \times G$ and ι_i denotes the canonical injection into the i-th component of $H \times H$. By hypothesis all $(f_{ij}(H) + H)/H$ ($i, j \in \{1, 2\}$) are finite. Then $f(H \times H) \subseteq (f_{11}(H)+f_{21}(H))\times(f_{12}(H)+f_{22}(H))$. Since $(f_{11}(H)+f_{21}(H)+H)/H = (f_{11}(H)+H)/H + (f_{21}(H)+H)/H$ and $(f_{12}(H)+f_{22}(H)+H)/H$ are finite, it follows that $(f(H \times H) + H \times H)/H \times H$ is also finite. □

Remarks. 1) The previous statement cannot be extended to infinite direct sums or direct products. To see this, it is enough to consider the case $H = \mathbf{Z}$ and $G = \mathbf{Q}$. Since $\mathbf{Q}$ is a epimorphic image of $\mathbf{Z}^{(\omega)}$ and of $\mathbf{Z}^{\omega}$, it is easy to construct homomorphisms $\mathbf{Z}^{(\omega)} \to \mathbf{Q}^{(\omega)}$, respectively $\mathbf{Z}^{\omega} \to \mathbf{Q}^{\omega}$, which do not verify the condition in the definition of strongly inert subgroups.

2) If H and K are strongly inert subgroups of G, $H \oplus K$ might not be strongly inert in $G \oplus G$. Indeed, take (as in [7, Example 2.5]) two non-isomorphic non-zero rational groups A and B. Then both are strongly inert in $\mathbf{Q}$ but $A \oplus B$ is not even fully inert in $\mathbf{Q} \oplus \mathbf{Q}$.

Observe that this differs from the property proved above: this happens if one takes the direct sum of two different (not isomorphic) subgroups.

The similar property holds also for strongly invariant subgroups.

SI7) *If $K \leq H \leq G$ and K is strongly inert in a group G, then H may not be strongly inert in G.*

Obviously, for $K = 0$ one can take any non strongly inert subgroup in a group G. We give now a non-trivial

Example. For an infinite set I consider $G = N \oplus L = (\bigoplus_{i \in I} \mathbf{Z}(4)) \oplus (\bigoplus_{i \in I} \mathbf{Z}(4))$ and take $K = G[2]$ the socle, which is strongly invariant in G and so strongly inert.

The subgroup $H = N + G[2]$ is not strongly inert. Indeed, take the isomorphism $t : N + G[2] \longrightarrow L + G[2]$ composed with the inclusion $\iota_{L+S(G)}$ in G, i.e., $f = \iota \circ t : H \longrightarrow G$. Then $f(H) + H = G$ and G/H is not finite.

Again this is also an Abelian example for **SI7** in [4] (where the given example was the not commutative quaternion 8-group).

Notice that if H and K are commensurable the property holds true

LEMMA 3.5. *Let $K \leq H$ be subgroups in a group G and H/K is finite. If K is strongly inert in G then H is strongly inert in G.*

PROOF. Suppose $f : H \longrightarrow G$ is a homomorphism, and consider the restriction $f|_K : K \longrightarrow G$. Then $\frac{K+f(K)}{K}$ is finite and we must show that $\frac{H+f(H)}{H}$ is finite. Consider the canonical surjective homomorphism $\pi : \frac{H+f(H)}{K} \longrightarrow \frac{H+f(H)}{H}$. Since $\ker \pi = H/K$ is finite, $\frac{H+f(H)}{K}$ is finite iff $\frac{H+f(H)}{H}$ is finite.

Since H/K is finite, also $f(H)/f(K)$ is finite, and so $\frac{H+f(H)}{K+f(K)}$ is finite as well. Since $\frac{H+f(H)}{K+f(K)}$ is isomorphic to $\frac{\frac{H+f(H)}{K}}{\frac{K+f(K)}{K}}$ and $\frac{K+f(K)}{K}$ is finite by assumption, $\frac{H+f(H)}{K}$ is also finite. By what was observed above, $\frac{H+f(H)}{H}$ is finite, as desired. □

The similar property fails also for strongly invariant subgroups.

SI8) *If $K \leq H \leq G$ and H is strongly inert in a group G, then K may not be strongly inert in G.*

This is witnessed by the example in **SI1**.

The similar property fails also for strongly invariant subgroups.

SI9) *If $K \leq H \leq G$ and H is strongly inert in a group G, then H/K may not be strongly inert in G/K.*

In order to give an example first observe that from Proposition 3.1 (b), we immediately get: *if $A \cong B$ are infinite (sub)groups then A is not strongly inert in $A \oplus B$.*

Example. For an infinite set I consider $G = H \oplus K = (\bigoplus_{i\in I} \mathbf{Z}(2)) \oplus (\bigoplus_{i\in I} \mathbf{Z}(4))$. Then the socle $G[2] = H + 2K$ is strongly invariant, and so strongly inert in G. However $(H+2K)/2K \cong H = (\bigoplus_{i\in I} \mathbf{Z}(2))$ is, according to the previous observation, not strongly inert in $G/2K \cong (\bigoplus_{i\in I} \mathbf{Z}(2)) \oplus (\bigoplus_{i\in I} \mathbf{Z}(2))$.

The similar property fails also for strongly invariant subgroups.

4. Strongly inert subgroups

In [7], [8], and [11], the fully inert subgroups of free groups, p-groups, respectively divisible groups were studied and described.

The determination of strongly inert subgroups of divisible groups is simple. Indeed, since divisible groups are injective, every homomorphism from a subgroup can be extended to an endomorphism of the whole group and so a subgroup is strongly inert in a divisible group iff it is fully inert (see also Example (a) in Section 2).

Strongly inert subgroups of free groups are characterized by the following

THEOREM 4.1. *A subgroup H of a free group G is strongly inert iff the index $[G : H]$ is finite.*

PROOF. Since the finite index condition is clearly sufficient, suppose G is a free group and H is strongly inert in G. Then G/H is torsion and since H is also free of the same rank as G, there exists an isomorphism $\varphi : H \longrightarrow G$. Since $G/H = (\varphi(H) + H)/H$ is finite, H has finite index in G. □

In the literature on fully inert subgroups, a central theme is whether fully inert subgroups are (or not) characterized by the property of being commensurable with a fully invariant subgroup. An important result, which follows easily from the fact that intersections of fully inert subgroups are fully inert, is that if a subgroup is commensurable with a fully inert subgroup, it is itself fully inert (see Corollary **2.9** in [7]).

Since intersections of strongly inert subgroups need not be strongly inert, the situation is different in the strongly inert context (as it also was in the strongly invariant case): for the analogue result we need a completely different proof. The remaining goal of this section is to see when a subgroup commensurable with a strongly invariant subgroup is strongly inert and vice-versa.

As for classes of torsion-free groups, this study depends on our knowledge of strongly invariant subgroups of torsion-free groups. We first recall (Proposition 26 in [4]) that *fully transitive homogeneous groups of idempotent type are strongly invariant simple* (i.e., have no proper strongly invariant subgroups). Here, a torsion-free group is fully transitive if for any two of its elements a, b with characteristics $\chi(a) \leq \chi(b)$, there exists an endomorphism of the group sending a upon b, and is homogeneous if all the elements $\neq 0$ are of the same type. This is a large class containing (homogeneous) separable and algebraic compact groups (of idempotent type).

It is readily seen that a subgroup H of a group G is commensurable with 0 iff it is finite, and commensurable with G iff it has finite index. In both cases, such subgroups are strongly inert. Therefore

COROLLARY 4.2. *Subgroups commensurable with strongly invariant subgroups of fully transitive homogeneous groups of idempotent type are trivially strongly inert.*

In closing, we deal with torsion groups. First notice an expected reduction

Theorem 4.3. *Let H be a subgroup in a torsion group G. Then H is strongly inert in G iff H_p is strongly inert in G_p for every prime p, and almost all primary components are strongly invariant.*

Proof. ($\Rightarrow$) For an arbitrary prime p, let $H_p \xrightarrow{f_p} G_p$ be a group homomorphism. Since H_p is a direct summand of H, we write $H = H_p \oplus K$, and we trivially extend f_p to an $f : H \longrightarrow G_p$ and further to $f : H \longrightarrow G$ such that $f(K) = 0$. Therefore, $(f(H) + H)/H = (f_p(H_p) + H)/H$ is finite. Since $f_p(H_p) \leq G_p$, the latter is $[f_p(H_p) + (H_p \oplus K)]/(H_p \oplus K)$ with $f_p(H_p) \cap K = 0$. Since both $f_p(H_p)$ and H_p are included in G_p and $G_p \cap K = 0$, $(f_p(H_p) + H_p)]/H_p$ embeds in the finite group $[f_p(H_p) + (H_p \oplus K)]/(H_p \oplus K)$. Then $(f_p(H_p) + H_p)]/H_p$ is finite.

Suppose that there exists infinitely many primes p such that H_p is not strongly invariant in G_p, and denote by $\mathcal{S}$ the set of these primes. For every prime $p \in S$ there exists a homomorphism $f_p : H_p \to G_p \hookrightarrow G$ such that $f_p(H_p)$ is not contained in H_p. If $p \notin \mathcal{S}$ we will denote by $f_p : H_p \to G_p$ the zero homomorphism. Then for $f = \oplus_p f_p : H \to G$, the quotient $[f(H) + H]/H \cong \oplus_p [f_p(H_p] + H_p]/H_p$ is infinite, a contradiction. Therefore, for almost all primes p, H_p is strongly invariant in G_p.

($\Leftarrow$) Conversely, let $H \xrightarrow{f} G$ be a group homomorphism. Since we can decompose it into $H = \bigoplus_p H_p \xrightarrow{\oplus f_p} \bigoplus_p G_p = G$ and by hypothesis, $(f_p(H_p) + H_p)/H_p$ are nonzero but finite only for finitely many p-components, finally, $(f(H) + H)/H \cong \bigoplus_p (f_p(H_p) + H_p)/H_p$ is finite. ☐

Next recall from [11, (1.4)] the following

Proposition 4.4. *Let G be a p-group, and H, K two subgroups of G. Then H is commensurable with K iff $H = F \oplus C$ and $K = F' \oplus C$, where F and F' are finite, for some subgroup C of G.*

The following result was conjectured for reduced p-groups by P. Danchev.

Theorem 4.5. *Let G be a torsion group. A subgroup $H \leq G$ is strongly inert iff H is commensurable with a strongly invariant subgroup of G.*

Proof. ($\Rightarrow$) Suppose that H is strongly inert in G. By Theorem 4.3, H_p is strongly invariant in G_p for almost all primes p.

Thus, it suffices to prove that for every p-component H_p which is not strongly invariant in G_p there exists a non-negative integer n such that H_p is commensurable with $G_p[n]$. Recall from Theorem 4.3 that all such primary components H_p are strongly inert in G_p.

Let p be a prime such that H_p is not strongly invariant in G_p. We have three cases:

(I) H_p is reduced, but not bounded. Suppose H_p is not of finite index in G_p. Then there exist countably many elements $x_n \in G_p \setminus H_p$, $n \in \mathbb{N}$. Since H_p is unbounded and reduced there exists an epimorphism $f : H_p \to \langle x_n \mid n \in \mathbb{N}\rangle$ and so H_p is not strongly inert, a contradiction. Therefore H_p is of finite index in G_p, so H_p is commensurable to $G_p[0]$.

(II) H_p is bounded. Let k be the smallest positive integer such that $p^k H_p = 0$. So we have a direct decomposition

$$H_p = \bigoplus_{\ell=1}^{k} \left(\bigoplus_{i \in I_\ell} \langle h_i \rangle \right),$$

where I_ℓ are some sets and for every $i \in I_\ell$ the order of h_i is p^ℓ.

Suppose H_p is not commensurable to 0 (i.e. H_p is not finite) and that for every $\ell \in \{1, \dots, k\}$ the subgroup H_p is not commensurable to $G[p^\ell]$. Since H_p is infinite there exists a maximal integer $m \in \{1, \dots, k\}$ such that I_m is infinite. Then

$$\bigoplus_{\ell=1}^{m} \left(\bigoplus_{i \in I_\ell} \langle h_i \rangle \right)$$

is not of finite index in $G[p^m]$, hence there exist infinitely many elements $x_j \in G[p^m] \setminus H_p$, $j \in J$, of order $\leq p^m$ such that $\langle x_j \mid j \in J\rangle = \bigoplus_{j \in J} \langle x_j \rangle$ is a direct summand of $G[p^m]$. Using the fact that I_m is infinite, we can apply the same technique as in (I) to obtain a homomorphism $f : H_p \to G_p$ such that H_p is of infinite index in $f(H_p) + H_p$. Then the set I_m must be finite, a contradiction.

(III) H_p in not reduced. Let $E_p \neq 0$ be the divisible part of H_p. Every infinite cocyclic subgroup of G_p is an epimorphic image of H_p. Since infinite cocyclic groups have no finite quotients, it follows that E_p is the divisible part of G_p. Moreover, if $G_p = E_p \oplus R_p$ with R_p reduced, we have $H_p = E_p \oplus K_p$, where $K_p = H_p \cap R_p$. It is easy to see that H_p is strongly inert in G_p iff K_p is strongly inert in R_p. By the first two cases it follows that K_p is commensurable to $R_p[n]$ for some non-negative integer n and the proof is complete.

($\Leftarrow$) Let $f : H \to G$ be a homomorphism and let K be a strongly invariant subgroup of G such that H is commensurable with K. Then we have $H_p = K_p$ for almost all p.

For every prime p such that $H_p \neq K_p$, by Proposition 4.4, there exists a subgroup $C_p \leq H_p \cap K_p$ such that we have direct decompositions $H_p = F_p \oplus C_p$, $K_p = F'_p \oplus C_p$, with finite subgroups F_p, F'_p. If $f_p : H_p \to G_p$ is the homomorphism induced by f, then $f(C_p) \subseteq K_p$, hence $f_p(H_p) + H_p \subseteq f(F_p) + C_p + F'_p + F_p = f(F_p) + F'_p + H_p$, and it follows that H_p is of finite index in $f_p(H_p) + H_p$. Applying Theorem 4.3, H is strongly inert in G. □

Thus, since (see [4, Proposition 12]) the only (reduced) strongly invariant subgroups of a reduced p-group G are the subgroups $G[p^n]$, for all positive integers n, the theorem above determines the strongly inert subgroups of torsion groups.

Finally observe that strongly inert subgroups need not be commensurable with strongly invariant subgroups. Indeed, all subgroups of $\mathbf{Q}$ are strongly inert, but only $\mathbf{Q}$ is strongly invariant in $\mathbf{Q}$.

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