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RESTRICTED SOCLE CONDITIONS IN LATTICES

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In 1974 Iacovlev found necessary and sufficient conditions for a lattice to be isomorphic to the lattice of all subgroups of a group. The same problem for abelian groups seems to be much more difficult so that meanwhile several authors (Head, Kertesz, Delany, Fritsche, Richter and Stern) have done an "abelian theory of lattices", i.e. have found reasonable conditions for notions and results of the abealian group theory to be extended in lattices.

The standard algebraic conditions (modularity, upper continuity or compactly generation) generally do not suffice for this purpose, so, for our concern — the neat elements — we propose new conditions which we call "restricted socle conditions" by analogy with the ring theory.

We refer to [$C-D$] and [$S1$] for the wellknown notions of essential element, pseudocomplement, atom, socle, upper continuity, compact element, compactly generation, artinian lattice and modular lattice. We call a lattice L *atomic* if for every $0 \neq a \in L$ the quotient sublattice $a/0$ contains atoms, *torsion* [D] (and independently [C] !) if for every $1 \neq a \in L$ the quotient sublattice $1/a$ contains atoms and *strongly atomic* if for every $a < b$ the quotient sublattice b/a contains atoms. An element $n \in L$ is called *neat* [D] if n has a complement in $s/0$ for every atom s from $1/n$. If a is an element in an upper continuous lattice L then the set of all the essential extensions of a is inductive and contains (by Zorn's lemma) maximal elements, called *maximal essential extensions* of a; the element a is called *essential closed* if a has no essential extension $\neq a$ in L.

1. THEOREM. *Each maximal essential extension of an element (in particular, each essential closed element) is a neat element.*

Proof. If x is not neat in L we show that x is not a maximal essential extension. From our hypothesis, there is an atom s in $1/x$ such that x has no complement in $s/0$. We prove that x is essential in $s/0$. If x were not essential in $s/0$, there would be an element $t \neq 0$, $t \in s/0$ such that $x \wedge t = 0$. In this case $x_{\neq} < x \vee t$ and $t \leqslant s$ implies $x \vee t \leqslant x \vee s = s$ so that $x \vee t = s$ (s atom in $1/a$). But t is now a complement of x in $s/0$, contradiction. So, x is not a maximal essential extension.

2. *Remarks.* For abelian groups, this result is due to T. Szele.

It is known that in a pseudocomplement lattice, an element is a pseudocomplement if and only if it is essential closed, so that a part of this result can be derived from [D, theorem 12].

In order to get the converse of this result we consider the following "restricted socle" conditions :

(R1) For each $a < b$, a essential in $b/0 \Rightarrow b/a$ contains atoms;
(R2) For each $a < b$, a essential in $L \Rightarrow b/a$ contains atoms;
(R3) For each $a \neq 1$, a essential in $L \Rightarrow 1/a$ contains atoms.

3. *Remarks.* If L is a lattice with 0 and 1 the following implications are obvious : artinian $\Rightarrow$ strongly atomic $\Rightarrow$ R1 $\Rightarrow$ R2 $\Rightarrow$ R3.

If A is an abelian group the lattice of all the subgroups $L(A)$ satisfies ($R1$).

In [C] we proved that in a modular compactly generated lattice the condition ($R3$) is equivalent with : (a essential in $L \Leftrightarrow s(L) \leqslant a$ and $1/a$ is a torsion sublattice).

The following two conditions (taken from [L] and [$S2$]) are sufficient for a ring R in order to satisfy the condition (R3) for the lattice of all the submodules of an arbitrary R-module :
(L) any nonzero left ideal of R can be written as the product of a finite number of maximal left ideals of R;
(S) R is a commutative, noetherian ring such that every prime ideal is maximal.

4. THEOREM. *If L is a lattice with ($R1$) every neat element is a maximal essential extension. Moreover, every neat element is essential closed.*

Proof. We check that if x is not essentially closed then x is not neat. In this case there is an element $b \in L$, $x < b$ such that x is essential in $b/0$. Using ($R1$) b can be chosen as an atom in $1/x$. Now, if x would be neat in L, x should have a complement y in $b/0$ that is $b = x \vee y$ and $0 = x \wedge y$. But x is essential in $b/0$ so that $y = 0$ and $b = x$, contradiction. Hence x is not neat in L.

Remark. The equivalence "neat" — "essential closed" was established for abealian groups in [L].

5. LEMMA. *Let p, a, b be elements in a modular lattice L. If $p \wedge a = p \wedge b = 0$ and $(p \vee a) \wedge (p \vee b) = p$ then $(a \vee b) \wedge p = 0$.*
The proof is an exercise.

6. LEMMA. *Let $a, b \in L$ such that $a < b$; a is essential in $b/0$ if and only if for every independent finite set $\{a, a_1, \ldots, a_n\}$ the set $\{b, a_1, \ldots, a_n\}$ is independent too.*

The proof can easily be adapted from [H].

7. THEOREM. *In a modular lattice with ($R2$) the following conditions are equivalent :*

(i) *b is a pseudocomplement of x;*

(ii) *$x \wedge b = 0$, $x \vee b$ is essential in L and b is neat in L.*

Proof. The condition $x \wedge b = 0$ holds by definition and $x \vee b$ essential in L is known from [S1]. If s is a maximal essential extension of b, the set $\{x, b\}$ being independent, using (6) we derive $x \wedge s = 0$

so that $b = s$ (b is a pseudocomplement of x). Hence b is essential closed and neat (using 1)). Conversely, suppose $x \wedge b = 0$, $x \vee b$ essential in L and b neat in L but b is not a pseudocomplement of x. Let $x < c$ and $x \wedge c = 0$. We observe that $c \nleqslant x \vee b$ because b, being a complement of x in $(x \vee b)/0$, is also a pseudocomplement. Using ($R2$), $((x \vee b) \vee c)/$ $/(x \vee b)$ contains atoms so, by modularity, $c/(c \wedge (c \vee b)) = c/b$ contains atoms. If c' is an atom in c/b, b has a complement in $c'/0$ (b is neat in L) that is, an element $y \in c'/0$ such that $b \vee y = c'$, $b \wedge y = 0$. From $y <$ $< c' \leqslant c$ and $y \wedge c = 0$ we obtain $x \wedge y = 0$. Then $(x \vee x) \wedge (b \vee y) =$ $= (x \vee y) \wedge c' = y \vee (x \wedge c') = y \vee 0 = y$ and using (5), $(x \vee b) \wedge y =$ $= 0$. But $x \vee b$ is essential in L, $y = 0$ and $c' = b$, contradiction. Hence b is a pseudocomplement of x.

8. *Remark. In a modular lattice with* (R3) *the only essential neat element is* 1. Indeed, using ($R3$), if $x \neq 1$, $1/x$ contains atoms.
If s is an atom in $1/x$ and x is neat in L, x has a complement in $s/0$, i.e. there is an element $y \in s/0$ such that $x \vee y = s$ and $x \wedge y = 0$. If x is essential in L, $y = 0$ and $x = s$. Hence $x = 1$.

9. THEOREM. *In a modular upper continuous lattice with* ($R3$) *an element* $a \in L$ *is absolutely complemented* (or completely complemented, see [D]) *if and only if* $a \wedge b = 0$ *and* b *neat in* L *implies* $a \vee b$ *neat in* L.

Proof. The condition is necessary without ($R3$) (see [D]). Conversely, let p be a pseudocomplement of a (being upper continuous L is also pseudocomplemented). Using (7), p is neat in L so by hypothesis $a \vee p$ is also neat in L. But $a \vee p$ is also essential in L, so the above remark implies $a \vee p = 1$. Hence a *is* absolutely complemented.

This result shows that in [D] theorem 24 is true not only in strong atomic lattices but also in ($R3$) ones.

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