A survey of Abelian groups with reduced endomorphism ring

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Abstract

A ring without nonzero nilpotent elements is called reduced. Abelian groups with reduced endomorphism ring are studied. Characterization is given in the torsion case and large classes of mixed or torsion-free groups are found. Partly the results are proved via groups whose endomorphism ring is Abelian, that is, groups which have only fully invariant direct summands.

1 Introduction

Throughout this paper, all rings are associative with identity and all groups are Abelian. A ring without nonzero nilpotent elements is called *reduced*, and a ring is called *Abelian* if every idempotent is central. It is easy to check that an idempotent is central in a ring if (and only if) it commutes with all nilpotents elements. This way, reduced rings are (somehow trivially) Abelian.

In this note, groups with reduced endomorphism ring are investigated. We obtain complete results for torsion groups, and satisfactory results for mixed groups and for torsion-free groups.

In finding many of our results a simple observation was used: mostly all known results on groups which have commutative endomorphism ring (see [14] or [9]) hold *verbatim* for groups with Abelian endomorphism ring. This way, the groups we are searching for have to be found among these.

The word reduced will be, as it is already traditionally, used twofold. Reduced rings were defined above and reduced groups are (Abelian) groups without divisible subgroups. Since all groups we consider are Abelian, the term Abelian is used in the sequel only for rings.

Warning! *While finishing this note, the authors became aware (via [4], which was also difficult to discover, because of the term "normal" instead of Abelian) of the existence of [6], a paper published in a journal (Abelian Groups and Modules, Tomsk Univ., Tomsk) which does no more exist, and which was not included in Mathematical Reviews nor in Zentralblatt fur Mathematik. Moreover, this paper ([6]) was not included in the exhaustive bibliography of [9].*

Finally Professor Krylov sent us a copy of this (impossible to find!) paper. We make a necessary restitutio by listing the English translation of the results of [6], as the last Section.

2 Preliminaries

First notice that reduced endomorphism ring for modules is a property inherited by direct summands, and preserved under split extensions of fully invariant direct summands.

Further, recall that modules with Abelian endomorphism ring were characterized in [3] (or [4]), as modules having only fully invariant direct summands. However for our purposes we can do better

Proposition 1 *Let M be an R–module. The endomorphism ring $End(M)$ is reduced if and only if for every submodule $L \leq M$, $\mathrm{Hom}(M/L, L) = 0$.*

Proof. Suppose that $L \leq M$ is a submodule and let $f : M/L \to L$ be a homomorphism. If $\iota : L \to M$ is the inclusion map and $\pi : M \to M/L$ is the canonical surjection then $\iota f \pi$ is an endomorphism of M such that $(\iota f \pi)^2 = 0$. Since the endomorphism ring is reduced, $\iota f \pi = 0$, and so $f = 0$. Conversely, suppose $\mathrm{End}(R)$ is not reduced. Then there exists an endomorphism $f \neq 0$ of M such that $f^2 = 0$. If K is the kernel of f then $M/K \cong f(M) \subseteq K$, and we obtain a nonzero element in $\mathrm{Hom}(M/L, L) = 0$. ■

To have an easy reference we record here

Corollary 2 *For an R-module $M = L \oplus K$, $\mathrm{End}(M)$ is reduced if and only if both $\mathrm{End}(L)$ and $\mathrm{End}(K)$ are reduced and $\mathrm{Hom}(L, K) = 0 = \mathrm{Hom}(K, L)$.*

3 Torsion groups

In this section we characterize the torsion groups whose endomorphism ring is reduced. Since both are interesting, we give two different proofs for this characterization.

1) As noticed in the introduction, mostly all known results on groups which have commutative endomorphism ring (Szele, Szendrei [14]) hold *verbatim* for groups all whose direct summands are fully invariant. This way

Proposition 3 *Let G be a group with Abelian endomorphism ring. Then the p-components are indecomposable and $G/T(G)$ is divisible by all relevant primes.*

and

Theorem 4 *Let G be a torsion group. The following conditions are equivalent*

(1) $End(G)$ is commutative;

(2) $End(G)$ is Abelian;

(3) $G \cong \bigoplus C_p$, where C_p is a cocyclic group and p runs over some set of different primes.

Thus torsion groups with reduced endomorphism ring must be located among the torsion groups given in (3). Since $\mathrm{End}(G) \cong \prod \mathrm{End}(C_p)$, where either $\mathrm{End}(C_p) \cong \mathbf{Z}(p^k)$ for some k or $\mathrm{End}(C_p) \cong \mathbf{Q}_p^*$, and, all rings $\mathbf{Z}(p^k)$, $k \geq 2$ are not reduced but the ring of p-adic integers is reduced (actually a commutative domain), our conclusion is

Proposition 5 *A torsion group G has reduced endomorphism ring if and only if each primary component of G is simple or rank 1 divisible, i.e., $G \cong \bigoplus \mathbf{Z}(p^{n_p})$ where p runs over some set of different primes and $n_p \in \{1, \infty\}$.*

Rephrasing, the torsion groups with reduced endomorphism ring have the following form

$$\bigoplus_{p \in S} \mathbf{Z}(p)) \oplus (\bigoplus_{q \in U} \mathbf{Z}(q^{\infty}))$$

with S and U disjoint sets of prime numbers (i.e., an elementary component and a divisible part, but indecomposable primary components).

Examples. $\mathbf{Z}(p^2)$ has Abelian but not reduced endomorphism ring. $\mathbf{Z}(n)$ has reduced endomorphism ring if and only if n is a square-free positive integer.

2) Rephrasing again

Proposition 6 *Let G be an torsion Abelian group. Then* $\mathrm{End}(G)$ *is regular if and only if every p-component of G is isomorphic to $\mathbb{Z}(p)$ or $\mathbb{Z}(p^{\infty})$.*

Proof. Suppose that $\mathrm{End}(G)$ is regular. It is well known that every Abelian p-group has a cocyclic direct summand. If G is not divisible then we can take this direct summand to be finite, and it follows that G is cocyclic. Now the conclusion follows from Corollary 2. The converse implication was justified above. ■

4 Torsion-free groups

All groups we consider in this section are (Abelian and) torsion-free.

There are plenty of groups with non-reduced commutative endomorphism ring (e.g., $\mathbf{Z}(n^i)$ for $n, i \geq 2$), and many noncommutative reduced rings (e.g., direct products of noncommutative domains).

Surprisingly, this is not the case for the large class of separable torsion-free groups.

In order to prove this we need another useful inclusion of the class of all reduced rings (as inclusion in the class of Abelian rings was used for torsion and mixed groups): every reduced ring is semiprime.

We first recall the definitions: the *prime radical* $P(R)$ of a ring R is the intersection of all prime ideals of R. A ring is *semi-prime* if $P(R) = 0$. Equivalently, a ring R is semi-prime if $aRa = (0)$ implies $a = 0$. A commutative ring is semiprime if and only if it is reduced.

Theorem 7 *For a separable torsion-free group G, the following conditions are equivalent*

(i) $\mathrm{End}(G)$ *is commutative;*

(ii) $\mathrm{End}(G)$ *is reduced;*

(iii) $G = \bigoplus_i H_i$ *where H_i are groups of rank 1, with pairwise incomparable types.*

Proof. The equivalence (i) $\Longleftrightarrow$ (iii) may be found in [9], Exercise 6 (p.132). We proceed by proving (ii) $\Longleftrightarrow$ (iii).

If $\mathrm{End}(G)$ is reduced, it is semi-prime, and so by Theorem **23.13** and Lemma **23.9** in [9], it is homogeneously decomposable, say $G = \bigoplus_{\mathbf{t}\in\Omega} G(\mathbf{t})$ (here Ω denotes the set of types of all direct summands of rank 1). As direct summand, each $\mathrm{End}(G(\mathbf{t}))$ must be reduced. Finally, it must be of rank 1. Otherwise, such a group G (see [5], p. 235) has no fully invariant subgroups other that nG, but these are not pure and so nor proper direct summands. Since the rank $r(G) > 1$, and the group is separable, G has proper direct summands. Thus, such groups have direct summands which are not fully invariant, a contradiction.

Conversely, the subgroups H_i are fully invariant and we use a split extension. ■

Again by [9], Exercise 6, for separable torsion-free groups, another two conditions are equivalent: $\mathrm{End}(G)$ is right *subcommutative* (R is right subcommutative if $Ra \subseteq aR$ for each $a \in R$) and/or, G is *stable* (every endomorphic image of G is fully invariant). This result has many important consequences.

Corollary 8 *(i) The endomorphism ring of any homogeneous separable group of rank at least two is not Abelian (and so) nor reduced.*

(ii) The endomorphism ring of any homogeneous completely decomposable torsion-free group of rank at least two is not Abelian (and so) nor reduced. In particular, so are free or divisible groups of rank at least two.

(iii) If the divisible part of a group G has at least rank two then $\mathrm{End}(G)$ *is not Abelian (and so) nor reduced.*

(iv) The only non reduced group with Abelian endomorphism group is $\mathbf{Q}$. *Thus if* $\mathrm{End}(G)$ *is reduced then $G \cong \mathbf{Q}$ or else G is reduced.*

Proof. (iv) Since only groups $\mathbf{Q}\oplus R$ (with possible $R = 0$) may have Abelian endomorphism ring, if $R \neq 0$ then $\mathrm{Hom}(R, \mathbf{Q}) \cong \prod_{r_0(R)} \mathbf{Q} \neq 0$ and so R is a direct summand which is not fully invariant. Thus $\mathrm{End}(\mathbf{Q}\oplus R)$ is not Abelian. ■

Since $\mathrm{End}(\mathbf{Q}) \cong \mathbf{Q}$ is a field, it is obviously reduced. Similarly, if R rational group (i.e., a subgroup of $\mathbf{Q}$) then $\mathrm{End}(R)$ is isomorphic to a subring of $\mathbf{Q}$ and so it is reduced. In particular, so is $\mathrm{End}(\mathbf{Z}) \cong \mathbf{Z}$.

Further, recall (see [5]) that a family of groups $\{G_i\}_{i\in I}$ is a *rigid system* if $\mathrm{Hom}(G_i, G_j) \cong \begin{cases} R & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$ with a subgroup R of $\mathbf{Q}$. That is, groups in a rigid system have no other endomorphisms then multiplications by rational numbers. In particular, these have no idempotent endomorphisms $\neq 0$ and 1, thus they are necessarily indecomposable. Moreover, endomorphism rings of groups in a rigid system are commutative *and reduced* (no nonzero nilpotent endomorphisms).

Example. Any set of rank 1 groups with incomparable types.

Therefore, as for commutative endomorphism rings, *rigid systems provide also reduced endomorphism groups.*

In determining groups with commutative endomorphism ring, indecomposable and especially strongly indecomposable groups play a crucial rôle. This does not happen for reduced endomorphism rings, where (even) stronger conditions must be imposed.

Just looking at (iii) in Theorem 7 we conclude

Proposition 9 *There exist groups with reduced endomorphism ring which are decomposable.*

As for the converse, more can be proved

Proposition 10 *For every positive integer n there exists a strongly indecomposable group of rank n whose endomorphism ring is not reduced.*

Proof. In [1] it is proved that, if G is strongly indecomposable, then every endomorphism of G is a monomorphism or else nilpotent. More, an example is given (Example 2.5, p. 24) of strongly indecomposable group of rank n with a non-zero nilpotent endomorphism. ■

Example. Again from [1]: there are strongly indecomposable groups of rank 2 whose quasi-endomorphism ring $Q\mathrm{End}(G)$ is isomorphic to the ring of 2×2 triangular matrices $\left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} | a, b \in \mathbf{Q} \right\}$. It is readily checked, since $Q\mathrm{End}(G)$ is Artinian, that the Jacobson radical is nilpotent. Actually all $\left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} | b \in \mathbf{Q} \right\}$ are nilpotent. Therefore $Q\mathrm{End}(G)$ and $\mathrm{End}(G)$ are not reduced.

Further, recall that a torsion-free group was called *irreducible* (by J. Reid) if it has no proper pure fully invariant subgroups.

Proposition 11 *Any irreducible group with Abelian endomorphism ring is indecomposable.*

Proof. Such a group has only fully invariant summands. Being irreducible, it has only trivial direct summands. ■

As a special case, any irreducible group with reduced endomorphism ring is indecomposable. Clearly, the previous result holds for a larger class of groups (with Abelian or reduced endomorphism ring): those who have no nontrivial fully invariant summands.

Quasi-endomorphisms may also be used related to our subject.

For finite rank torsion-free groups we have the following:

Theorem 12 *Let G be a finite rank torsion-free group. Then* $\mathrm{End}(G)$ *is reduced if and only if G is quasi-isomorphic to direct sum $A_1 \oplus \ldots \oplus A_k$ of torsion-free abelian groups of finite rank whose quasi-endomorphism rings are division rings and* $\mathrm{Hom}(A_i, A_j) = 0$ *for all $i \neq j$.*

Proof. Let G be quasi-isomorphic to $A_1 \oplus \ldots \oplus A_k$ such that all groups A_i are strongly indecomposable. We can suppose that $nG \leq A_1 \oplus \ldots \oplus A_k \leq G$ for some integer $n > 0$. From $\mathrm{Hom}(G/A_i, A_i) = 0$ we deduce $\mathrm{Hom}(\bigoplus_{j \neq i} A_j, A_i) = 0$. Moreover, if f is a nilpotent endomorphism for A_i then it can be viewed as a nilpotent endomorphism of $A_1 \oplus \cdots \oplus A_k$. Since $nf_{nG}(nG) \subseteq nG$, it follows that $nf(nG) = 0$, and this is possible only if $f = 0$. Then A_i has no nilpotent quasi-endomorphisms. But $\mathbb{Q}\mathrm{End}(A_i)$ is a local finitely generated $\mathbb{Q}$-algebra, hence $\mathbb{Q}\mathrm{End}(A_i)$ is a division ring. ■

This result (with another proof) appears also in [6] (but we found it only in [2]), where it is also proved (see also [7]) *that if a torsion-free group of finite rank has reduced endomorphism ring, the quasi-decomposition above is unique.* Moreover, for such groups G, $\mathrm{End}(G)$ *is reduced if and only if* $G = \mathrm{Soc}(G)$ *and all A_i are fully invariant in G, or, if and only if* $G = \mathrm{Soc}(G)$ *and all A_i form a rigid system.* Here the pseudo-socle $\mathrm{Soc}(G)$ of a torsion-free group is the pure subgroup generated by all minimal pfi (pure and fully invariant) subgroups.

Notice that actually, for torsion-free groups of finite rank $G = \mathrm{Soc}(G)$ if and only if $\mathrm{End}(G)$ is semi-prime (see [9]).

5 Mixed groups

Needless to say, the mixed groups we consider in the sequel are genuine (i.e., $0 \neq T(G) \neq G$).

Again, locating the reduced endomorphism rings among the Abelian endomorphism rings is useful.

We first deal with the splitting mixed case.

Since (see [9])

Theorem 13 ***(i)*** *Let $G = T(G) \oplus F$ a splitting mixed group. The ring $End(G)$ is Abelian if and only if $T(G) = \bigoplus_{p \in S} G_p$, with a set of primes S, $G_p \cong \mathbf{Z}(p^{k_p})$, k_p is a positive integer, $pF = F$ for all $p \in S$ and the ring $End(F)$ is Abelian.*

(ii) *If G is a mixed group and $End(G)$ is Abelian then $T(G) \cong \bigoplus_{p \in S} \mathbf{Z}(p^{k_p})$, k_p is a positive integer and $p(G/T(G)) = G/T(G)$. If $H = \{a \in G | h_p(a) = \infty$ for all $p \in S\}$ then H is a torsion-free subgroup of G such that $pH = H$ for all $p \in S$.*

Clearly, we can replace "Abelian" with "reduced" in **(ii)**. Thus, mixed groups with reduced endomorphism ring have finite indecomposable (i.e., cocyclic) primary components and so

Corollary 14 *Mixed groups with reduced endomorphism ring have reduced torsion part. Moreover, the torsion part has finite indecomposable (i.e., cocyclic) primary components. As such (e.g.,* ***27.1*** *[5]), the primary components are direct summands.*

It is now easy to discard the splitting mixed case.

Proposition 15 *A splitting mixed group* $G = T(G) \oplus F$ *has a reduced endomorphism ring if and only if if* $T(G) = \bigoplus_{p \in S} G_p$*, with a set of primes* S*,* $G_p \cong \mathbf{Z}(p)$*,* $pF = F$ *for all* $p \in S$ *and the ring* $End(F)$ *is reduced.*

Proof. Since reduced endomorphism ring is a property inherited by direct summands, the conditions are clearly necessary according to **(i)** and Proposition 5. Conversely, again by **(i)**, $\mathrm{End}(G)$ is Abelian and so both $T(G)$, F are fully invariant (and have reduced endomorphism ring). Hence the claim. ∎

However, an infinite direct product $\prod_{p \in P} \mathbf{Z}(p)$ for a set of primes P, shows that groups with reduced endomorphism rings are generally not splitting.

Further, using again Corollary 2 (see also Corollary 8, (iv)), we can solve the non-reduced mixed case.

Corollary 16 *Suppose* $G = D \oplus R$ *is a mixed non-reduced group with divisible* D *and reduced* R*. Then* $\mathrm{End}(G)$ *is reduced iff* $G = T \oplus \mathbb{Q}$*, where* T *is a reduced torsion group with* $\mathrm{End}(T)$ *reduced.*

As for reduced mixed groups, we were not able to improve a result from [4] (again reduced endomorphism rings are accessed via Abelian endomorphism rings) and its Corollary, that is

Theorem 17 *Let* G *be a reduced mixed group with Abelian endomorphism ring,* $S(G)$ *the set of all primes* p *such that* $T_p(G) \neq 0$ *(all primes which are relevant for* G*),* $B = \bigcap_{p \in S(G)} p^\omega G$ *and* $H = \overline{T(G)}$ *the* $\mathbf{Z}$*-adic closure of* $T(G)$*. Then* $H \cap B = 0$ *and* $H \oplus B$ *is a pure fully invariant subgroup of* G*, the endomorphism ring* $\mathrm{End}(A)$ *is commutative and* A *is representable as*

$$\bigoplus_{p \in S(G)} H_p \leq H \leq \prod_{p \in S(G)} H_p = S$$

where $H_p = G_p$ *are cyclic* p*-groups and* H *is pure in* S*.*

Corollary 18 *If* G *is a group with reduced endomorphism ring, then a subgroup* H *of* G *has commutative endomorphism ring,* H_p *are cyclic groups of prime order, and the quotient* G/H *is, up to isomorphism, representable as*

$$\bigoplus_{p \in S(G)} H_p \leq H \leq \prod_{p \in S(G)} H_p = S$$

where $H_p \cong G_p$*, and moreover, the subgroup* G/H *is* p*-pure in* S *for every* $p \in S(G)$*.*

6 Kozhukhov results

Lemma 1. *The endomorphism ring of the group* $G = \bigoplus_{i \in J} G_i$ *is reduced iff (a)* $\mathrm{End}(G_i)$ *are reduced, for all* $i \in J$*; (b) the system id rigid (i.e.,* $\mathrm{Hom}(G_i, G_j) = 0$ *for every* $i, j \in J$*).*

Theorem 2. *The endomorphism ring of a torsion group is reduced iff each* p*-component is isomorphic to* $\mathbf{Z}(p)$ *or to* $\mathbf{Z}(p^\infty)$*.*

From Fuchs

Lemma 3. *For an arbitrary Abelian group* G*,* $\mathrm{Hom}(G, \mathbf{Z}(p)) = 0$ *iff* G *is* p*-divisible.*

Lemma 4. $\mathrm{Hom}(G, \mathbf{Z}(p^\infty)) \neq 0$ *whenever* G *is mixed.*

Then [with proofs]

Corollary 5. *If* $\mathrm{End}(G)$ *is reduced, for a mixed group* G *then the torsion part* $T(G) = \bigoplus_p \langle g_p \rangle$ *with* $o(g_p) = p$*.*

Lemma 6. *If* $\mathrm{End}(G)$ *is reduced, for a mixed group* G *then* $G/T(G)$ *is* p*-divisible for all relevant primes* p*.*

Lemma 7. *If* $\mathrm{End}(G)$ *is reduced, for a mixed group* G *then for every coset* $g + T(G)$ *there exists an element* a *such that* $h_p^G(a) = h_p^{G/T(G)}(g + T(G))$ *for every prime* p*.*

Theorem 8. *The endomorphism ring of a mixed group* G *is reduced iff it splits* $G = T(G) \oplus F$ *and the following conditions are fulfilled:*

(a) every p*-component is isomorphic to* $\mathbf{Z}(p)$*;*

(b) $\mathrm{End}(F)$ *is reduced;*

(c) $pF = F$ *for every relevant prime* p*.*

Further torsion-free groups are discussed.

Lemma 9. *Let* G *be torsion-free and let* H *be a quasi-equal subgroup (*$nG \subset H$*). If* A *is a (minimal) pfi-subgroup of* G*, then* $\langle nA \rangle_*^H$ *is also (minimal) pfi-subgroup of* H*. Conversely, if* A *is a (minimal) pfi-subgroup of* H*, then* $\langle A \rangle_*^G$ *is a (minimal) pfi-subgroup of* G*.*

Corollary 10. *Let* G *be a torsion-free group and* $A \oplus B$ *a quasi-equal subgroup. If* A *is pure in* G *and* $\mathrm{Hom}(A, B) = 0$ *then* A *is fully invariant in* G*.*

In [10], it is proved that if $\mathrm{End}(G)$ is reduced for a tf group of finite rank then $G = \mathrm{Soc}(G)$.

The author uses the class $\mathcal{A}$ of groups (see Reid [13])

Lemma 11. *If* $G \in \mathcal{A}$ *has reduced endomorphism ring, then* $G = \mathrm{Soc}(G)$*.*

Theorem 12. Let $A = \bigoplus_{i \in J} A_i$ *be a (complete) quasi-decomposition of a group* $G \in \mathcal{A}$*. The following conditions are equivalent:*

a) $\mathrm{End}(F)$ *is reduced; b) for every* $i, j \in J$*,* $\mathrm{Hom}(A_i, A_j) = 0$ *and* $A_i = \mathrm{Soc}(A_i)$*; c)* $G = \mathrm{Soc}(G)$ *and for every* $i, j \in J$*,* $\mathrm{Hom}(A_i, A_j) = 0$*; d) for every* $i \in J$*,* A_i *is fully invariant in* G *and* $A_i = \mathrm{Soc}(A_i)$*; e)* $G = \mathrm{Soc}(G)$ *and or every* $i \in J$*,* A_i *is fully invariant in* G*.*

Corollary 13. *If* $G \in \mathcal{A}$ *has reduced endomorphism ring, then every quasi-decomposable pure subgroup of* G *is fully invariant in* G*.*

Theorem 14. *If $G \in \mathcal{A}$ has reduced endomorphism ring, then is has a unique (complete) quasi-decomposition.*

Corollary 15. *Suppose $G \in \mathcal{A}$ has reduced endomorphism ring and let $A = \bigoplus_{i \in J} A_i$ be its complete quasi-decomposition. Then every pure quasi-decomposable subgroup B of G is the pure envelope of a family of some A_i.*

Corollary 16. *Every quasi-decomposable group with reduced endomrphism ring in $\mathcal{A}$ has reduced endomorphism ring and belongs to $\mathcal{A}$* [?]

Corollary 17. *If $G \in \mathcal{A}$ has reduced endomorphism ring, then every (direct) quasi-decomposition in pure quasi-summands can be extended to a complete quasi-decomposition [?].*

The proofs of the last two theorems are defered to [8] (and use the fact that $\mathrm{End}(G)$ is reduced iff $\mathrm{End}(A)$ is reduced)

Theorem 18. *Let A be a subgroup of G, quasi-equal to G, which is either completely decomposable or is a vector group.* $\mathrm{End}(G)$ *is reduced iff all types in $\Omega(G)$ are incomparable and for every type $\alpha \in \Omega(G)$, $\tau_A(\alpha) = 1$.*

Theorem 19. *Let A be a separable subgroup quasi-equal to G. The endomorphism group of G is reduced iff A is completely decomposable and satisfies the conditions in Corollary 17.*

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