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PURE SUBGROUPS OF MIXED ABELIAN GROUPS INCLUDING THE TORSION PART

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1. Introduction

In what follows, for an abelian group G and an arbitrary subset $X \subseteq G$ we shall use the following notation $P(G,X)=\{g \in G | \exists n \in \mathbf{N}^*; ng \in \langle X \rangle\}$, that is, the elements that depend on X. We will study the elementary properties of these subgroups recovering several well-known results. These subgroups appear [2] in a particular case but no relevant use of them is made. Sometimes we shall denote by T the torsion part $T(G)$.

2. Elementary Results

LEMMA 2.1 *$P(G, X)$ is a pure subgroup of G.*

Indeed, if $g, h \in P(G, X)$ and $ng, mh \in \langle X \rangle$ for $n, m \in \mathbf{N}^*$ then $nm \in \mathbf{N}^*$ and $nm(g-h) \in \langle X \rangle$. Hence $g-h \in P(G, X)$. Further, if for $n \in \mathbf{N}^*$, $g \in nG \cap P(G, X)$, then there is an element $x \in G$ such that $g = nx$ and $m \in \mathbf{N}^*$ such that $mg \in \langle X \rangle$. So $mnx \in \langle X \rangle$ with $mn \in \mathbf{N}^*$ and hence $x \in P(G, X)$ and $g = nx \in nP(G, X)$. □

Obviously $X \subseteq Y \Rightarrow P(G, X) \leq P(G, Y)$. Moreover

LEMMA 2.2 *$P(G,\varnothing)=P(G,0)=T(G) \leq P(G,X)=P(G,\langle X \rangle) \leq P(G,G)=G$ and $T(G/\langle X \rangle)=P(G,X)/\langle X \rangle$.* □

LEMMA 2.3 *$P(G,X)=T(G) \Leftrightarrow X \subseteq T(G)$.*

Indeed, clearly $X \subseteq P(G,X)=T(G)$. Conversely, if $X \subseteq T(G)$ then $P(G,X) \subseteq \subseteq P(G,T(G))=T(G)$ (from $P(G,T(G))/T(G)=T(G/T(G))=0$). □

LEMMA 2.4 *$T(P(G,X))=T(G)$.*

Indeed, $P(G,X) \subseteq G \Rightarrow T(P(G,X)) \subseteq T(G)$; conversely $T(G) \subseteq P(G,X) \Rightarrow \Rightarrow T(T(G))=T(G) \subseteq T(P(G,X))$. □

LEMMA 2.5 *$P(G,X)=P(G,X \setminus T(G))$.*

Indeed, $P(G,X \setminus T(G)) \subseteq P(G,X)$ being clear, let $g \in P(G,X)$. If $ng \in \langle X \rangle$ with $n \in \mathbf{N}^*$ let $x_i \in T(G)$ and $y_i \in X \setminus T(G)$ such that $ng \in \Sigma_{i=1}^{m} x_i + \Sigma_{j=1}^{s} y_j$. For $t = \gcd(ordx_1, \ldots, ordx_m)$ we have $ntg \in \langle X \setminus T(G) \rangle$ with $nt \in \mathbf{N}^*$ so that $g \in P(G, X \setminus T(G))$. □

LEMMA 2.6 *If* $X \subseteq G$ *then* $P(G, X)$ *is the smallest pure subgroup of* G *which includes* X *and* $T(G)$.

Indeed, let H be a pure subgroup of G which contains X and $T(G)$ and $g \in P(G, X)$. Then $ng \in \langle X\rangle \subseteq H$ and $ng \in H \cap nG = nH$. There is an element $h \in H$ such that $n(g-h)=0$ and hence $g - h \in T(G) \subseteq H$. But so $g \in H$. □

Consequence 2.1 *Let us denote by* $S_P(G) = \{P(G, X) | X \in P(G)\}$. *The set of all the pure subgroups which contain* $T(G)$ *is exactly* $S_P(G)$.

Indeed, if P is a pure subgroup of G which contains $T(G)$ then $P(G, P \setminus T(G)) \subseteq P$ by the previous lemma. Conversely, $P = (P \setminus T(G)) \cup T(G) \subseteq P(G, P \setminus T(G))$ follows by **2.2** so that $P = P(G, P \setminus T(G)) = P(G, P)$ by **2.5.** □

Remark 2.1 *If* G *is a torsion group, the construction has no interest because* $P(G,X)=G$ *for each* $X \subseteq G$. *If* G *is torsion-free,* $P(G, X)$ *is the subgroup purely generated by* X *(the smallest pure subgroup which includes* X*). See also the next section.*

LEMMA 2.7 *If* A *is a subgroup of* G *then* $P(G,A) = A \Leftrightarrow G/A$ *is torsion-free.*
Indeed, this follows immediately from $P(G, A)/A = T(G/A)=0$. □
Now, it follows easily that

Consequence 2.2 G/A *torsion-free* $\Leftrightarrow$ A *is pure in* G *and includes* $T(G)$.

LEMMA 2.8 $\langle X\rangle$ *is essential in* $P(G, X) \Leftrightarrow S(G) \leq \langle X\rangle$.

Indeed, $P(G, X) / \langle X\rangle$ is a torsion group and $S(P(G,X)) = S(G) \cap P(G,X) =$

$= S(G) \leq \langle X\rangle$ because $S(G) \leq T(G) \leq P(G,X)$. □

Consequence 2.3 *In torsion-free groups* $\langle X\rangle$ *is essential in* $P(G, X)$ *for each subset* $X \subseteq G$.

PROPOSITION 2.1 *Let* $f: G \to H$ *a morphism of groups such that* $f \subseteq T(G)$. *Then* $P(f(G), f(a)) = f(P(G, a))$.

Obviously $ng \in \langle a\rangle \Rightarrow nf(g) \in \langle f(a)\rangle$ so that $f(P(G,a)) \subseteq P(f(G), f(a))$. Conversely, if $nf(g)=mf(a)$ for $n \in \mathbf{N}^*$ and $m \in \mathbf{Z}$ then $ng - ma \in \ker f \subseteq T(G)$.

Hence $k(ng-ma) = 0$ for a $k \in \mathbf{N}^*$ and so $(nk)g \in \langle a\rangle$ for $nk \neq 0$. □

The non-full subcategory $\mathcal{L}$ of **Ab** which consists only of the morphisms of groups $f: G \to H$ such that $\ker f \subseteq T(G)$ (i.e. vanishes on finite order elements) will be studied elsewhere. This could be a useful tool on mixed groups.

Consequence 2.4 $P(G/T, a+T) = P(G, a)/T$ *is true for each element of infinite order* a.

Indeed, the equality follows for $f: p_T: G \to G/T$ the canonic epimorphism.

PROPOSITION 2.2 *The subgroup $P(G, a)$ is isotype and $r_0(P(G, a))=1$ for each infinite order element $a \in G$.*

Indeed, by a well-known result of Megibben ([3]) the factor group $G / P(G,a) \cong (G / T) / (P(G,a) / T)$ is torsion-free like G/T because $P(G, a)/T$ is pure in G/T (see the previous consequence). For the second assertion, we have $r_0(P(G,a))=r(P(G,a)/T)=1$ because (again by the previous consequence) these are exactly all the elements (of the torsion-free group G/T) that depend of $a+T$. □

Consequence 2.5 *$G/P(G, a)$ is torsion-free and $r_0(G/P(G,a))+1=r_0(G)$.*

Consequence 2.6 *The factor group $P(G,a)/T$ is indecomposable (torsion-free).*

LEMMA 2.9 *$P(G, a)/T$ can be embedded in $\mathbf{Q}$ for each infinite order element a.*

Indeed ([4]), this is clear by the above proposition. In particular if G is a torsion-free group and $x \neq 0$ the function $f: P(G, x) \to \mathbf{Q}$ defined by $f(g) = \frac{m}{n}$ iff $ng=mx$ with $n \in \mathbf{N}^*$ is an embedding. □

PROPOSITION 2.3 *Each pure subgroup S of G which includes $T(G)$ and has torsion-free rank one has the form $P(G, a)$ for each $a \in S \setminus T(G)$.*

Indeed, if $a \in S \setminus T(G)$ then $P(G,a) \subseteq S$ follows by lemma **2.6.** Conversely, if $x \in S$ has infinite order ($x \in T(G) \Rightarrow x \in P(G,a)$ is obvious by lemma **2.2**) then $\{x, a\}$ is a dependent set (of infinite order elements) so that x depends on $\{a\}$ and $x \in P(G,a)$. □

Consequence 2.7 (*i*) $P(G,a)=G \Leftrightarrow r_0(G)=1$.

(*ii*) $\{S \leq G | T(G) \subseteq S, r_0(S) = 1, S \text{ pure in } G\} = \{P(G,a) | a \in G \setminus T(G)\}$. □

PROPOSITION 2.4 *If* $\operatorname{ord}(a) = \infty$ then $P(G,a) = \Sigma\{H \leq G | a \in H,\ r_0(H) = 1\}$.

One inclusion is obvious because $a \in P(G,a)$ and $r_0(P(G,a))=1$. As for the second, let $a, b \in H,\ r_0(H) = 1$. If $b \in T(G) \subseteq P(G,a)$ nothing is to be proved. If $\operatorname{ord}(b)=\infty$ then $\{a, b\}$ is dependent and hence $b \in P(G,a)$. □

We recall from [1] that a subset is called **pure-independent** if it is independent and generates a pure subgroup. Equivalently, $\{a_i\}_{i \in I}$ is pure-independent iff $mb=n_1a_1+..+n_ka_k$ implies $n_ja_j=mn'_ja_j$.

PROPOSITION 2.5 *$P(G,X)+P(G,Y) \leq P(G,X \cup Y)$ for every subsets X and Y of G; the equality holds if $X \cup Y$ is pure-independent.*

Proof. If $g=g_1+g_2$ with $g_1 \in P(G,X)$ and $g_2 \in P(G,Y)$ and $n_1g_1 \in \langle X \rangle$, $n_2g_2 \in \langle Y \rangle$ then $n_1n_2g = n_2(n_1g_1)+n_1(n_2g_2) \in \langle X \cup Y \rangle$ so that $g \in P(G,X \cup Y)$.

Conversely, if $g \in P(G,X \cup Y)$ and $ng = \Sigma_{i=1}^{s} n_ix_i + \Sigma_{j=1}^{t} m_jy_j$ with $n \in \mathbf{N}^*$, $X \cup Y$ being pure-independent, $n_ix_i = nn'_i x_i$ and $m_jy_j = nm'_j y_j$ and hence

$g = g_1 + g_2 + u$ where $g_1 = \Sigma_{i=1}^{s} n'_i x_i \in \langle X \rangle$, $g_2 = \Sigma_{j=1}^{t} m'_j y_j$ and $u \in T(G)$. Hence $g \in P(G,X) + P(G,Y)$ (indeed, $u \in T(G) \subseteq P(G,Y)$). □

Even for independent sets of infinite order elements, a characterization of pure-independence seems out of reach using the subgroups $P(G, X)$.

In the above-mentioned category $\mathcal{L}$ it is natural to consider the order epimorphism $\varphi_P : \mathcal{P}(G) \to S_P(G)$, $\varphi_P(X) = P(G,X)$.

If $f : G \to H$ is a morphism of groups such that $f \subseteq T(G)$ then the following diagram

$$\begin{array}{ccc} \mathcal{P}(G) & \xrightarrow{\varphi_G} & S_P(G) \\ \downarrow & & \downarrow \\ \mathcal{P}(H) & \xrightarrow{\varphi_{f(G)}} & S_P(f(G)) \end{array}$$

commutes, the vertical maps being induced by f.

PROPOSITION 2.6 $P(G, X \cap Y) \leq P(\langle X \rangle \cap \langle Y \rangle) = P(G,X) \cap P(G,Y)$.

Indeed, $P(G, X \cap Y) \leq P(G, \langle X \rangle) = P(G,X)$ and similarly for Y. Conversely, if $g \in P(G,X) \cap P(G,Y)$ and $ng \in \langle X \rangle$, $mg \in \langle Y \rangle$ then $nmg \in \langle X \rangle \cap \langle Y \rangle$ and hence $g \in P(G, \langle X \rangle \cap \langle Y \rangle)$. □

By the previous propositions we now derive

Remark 2.2 $(S_P(G), \subseteq)$ *is an upper directed lower semi-lattice.* □

3. **Relativization**

If H is a subgroup of G the construction we have studied in the previous section can be relativized. We use the same definition $P(H,X) = \{ h \in H \mid \exists n \in \mathbf{N}^* : nh \in \langle X \rangle \}$ where $X \subseteq G$. We have easy generalizations of the elementary results in the previous section (surely $P(H, X) = H \cap P(G, X)$).

LEMMA 3.1 $T(H) = P(H, \varnothing) = P(H,0) \subseteq P(H,X)$, $\langle X \rangle \cap H \subseteq P(H,X) \subseteq$ $\subseteq P(H,G) = H$ *and* $T(H / (\langle X \rangle \cap H)) = P(H,X) / (\langle X \rangle \cap H)$. □

LEMMA 3.2 $P(H,X) = T(H) \Leftrightarrow \langle X \rangle \cap H \subseteq T(H)$.

One implication being obvious, if $\langle X \rangle \cap H \leq T(H)$ one has to use the canonic epimorphism $H / (\langle X \rangle \cap H) \to H / T(H)$ to obtain $T(H / (\langle X \rangle \cap H)) = 0$ from $T(H/T(H)) = 0$ and hence $P(H, X) = T(H)$. □

LEMMA 3.3 $T(P(H, X)) = T(H)$. □

PROPOSITION 3.1 *$P(H, X)$ is the smallest pure (in G) subgroup of H which contains $\langle X \rangle \cap H$ and $T(H)$.*

Indeed, if K is a pure (in G) subgroup of H which contains $\langle X\rangle \cap H$ and $T(H)$ then $h \in P(H,X) \Rightarrow \exists n \in \mathbf{N}^*: nh \in \langle X\rangle \cap H \Rightarrow nh \in K \cap nG = nK \Rightarrow$ $\Rightarrow \exists k \in K: nh = nk \Rightarrow n(h-k) = 0 \Rightarrow h-k \in T(H) \subseteq K \Rightarrow h \in K$. □

PROPOSITION 3.2 *Let A be a subgroup of G.* $P(H,A) = H \cap A \Leftrightarrow H/(H \cap A)$ *is torsion-free.*

Indeed, this follows immediately from $P(H,A)/(H\cap A) = T(H/(H\cap A))$. □

Consequence 3.1 *If* A *is a subgroup of* H *then* $P(H,A) = A \Leftrightarrow H/A$ *is torsion-free.*

PROPOSITION 3.3 *Let P be an arbitrary pure subgroup of G. P=P(P, P) so that the relative construction gives all the pure subgroups from G.*

4. Non-standard Splitting

Following [4] we can consider the following class of mixed abelian groups: $G \in \mathcal{M}_3$ if G has non-trivial torsion-free direct summands. Moreover, we add the following classes: $G \in \mathcal{C}_1$ if there is an infinite order element $a \in G$ such that $P(G, a)$ is a direct summand of G and more generally $G \in \mathcal{C}_\alpha$ if there is a subset $X \subseteq G$, $X \cap T(G) = \varnothing$ card $X = \alpha$ such that $P(G, X)$ is a direct summand of G.

In this section we record some connections between the classes $\mathcal{C}_1$ and $\mathcal{M}_3$ and the class $\mathcal{S}$ of all the splitting mixed.

We first recall from [4] the following results:

LEMMA 4.1 *If* $G \in \mathcal{M}_3$ *with* $0 \neq B$ *torsion-free direct summand and G/T(G) is divisible then B is divisible too.*

For the proof, if B is a torsion-free direct summand, using a well-known decomposition of $T(G)$ (fully invariant subgroup of G), $T(G) = (T(G)\cap A) \oplus (T(G)\cap B)$ we derive $T(G) \subseteq A$ and so $G/A \cong (G/T(G))/(A/T(G))$ is divisible together with $G/T(G)$. Hence $B \cong G/A$ is divisible too. □

Consequence 4.1 *If for a reduced group G the factor group G/T is divisible then* $G \in \mathcal{M}_3$.

LEMMA 4.2 *If* $G \in \mathcal{M}_3$ *with* $0 \neq B$ *torsion-free direct summand and G/T(G) is indecomposable then G is splitting* (*moreover* $G = T(G) \oplus B$).

Indeed, with the above notations, $T(G)$ being fully invariant, one has $G/T(G) =$ $= ((A+T(G))/T(G)) \oplus ((B+T(G))/T(G))$. The factor group $G/T(G)$ being indecomposable we derive $B = 0$ or $G = T(G) \oplus B$. □

Consequence 4.2 *If for a non-splitting group G the factor group G/T is indecomposable then* $G \notin \mathcal{M}_3$.

Consequence 4.3 *If* $G \in \mathcal{C}_1$ *and G/T is indecomposable then G is splitting.*

LEMMA 4.3 *Every direct summand of a splitting mixed group is splitting. More precisely, if A is a direct summand of G and* $G = A \oplus B = T(G) \oplus H$ *then* $A = T(A) \oplus [(T(B)+H) \cap A]$. □

Consequence 4.4 *If* $G = A \oplus B$ *and A is non-splitting then G is non-splitting too.*

We continue with examples of mixed groups using the following notation: $\mathcal{A} = \mathcal{C}_1 \cap \mathcal{S}$ (here $\mathcal{S}$ is the set of all the splitting groups): if $G = T \oplus H = P(G,a) \oplus F$ using Lemma **2.4** and the above Lemma we obtain $P(G,a) = T \oplus (P(G,a) \cap H)$ and $G = T \oplus (P(G,a) \cap H) \oplus F$. Here $P(G,a) \cap H$ is a torsion-free group of rank 1. Moreover,

PROPOSITION 4.1 *Let F be a rank 1 torsion-free direct summand of G. For every* $a \in F$ *we have* $P(G,a) = T \oplus F$.

As direct summand of *G, F* and hence $T \oplus F$ is pure in *G* and contains *a*. But $r_0(T \oplus F) = r(F) = 1$ so that $P(G,a) = T \oplus F$ from the Proposition **2.3.** □

Consequence 4.5 *If G/T has a rank 1 direct summand S/T then S=P(G, a) for each* $a \in S \setminus T$.

Indeed, if we apply the above Proposition for the torsion-free group *G/T* we derive *P*(*G*/*T*, *a* + *T*) = *S*/*T* and *S*=*P*(*G*, *a*) from the Consequence **2.4.** □

Consequence 4.6 *Let* $G \in \mathcal{C}_1$ *(e.g.* $G = P(G,a) \oplus F$ *). Then G splits (i.e.* $G \in \mathcal{A}$ *) iff* $P(G,a) \in \mathcal{M}_3$ *(i.e. has torsion-free direct summands).*

If *G* splits from Lemma **4.3,** $P(G,a) \cap H$ is a torsion-free direct summand of *P*(*G*, *a*). Conversely, if $P(G,a) = K \oplus L$ for a torsion-free subgroup *L* then $1 = r_0(P(G,a)) = r_0(K) + r_0(L)$ so that $r_0(K) = 0$ and hence $K \subseteq T$. But $T \subseteq P(G,a)$ implies the equality *K*= *T*. □

Consequence 4.7 *If* $G \in \mathcal{C}_1$ *(e.g.* $G = P(G,a) \oplus F$ *) then G splits iff P(G,a) splits.*

From the above Consequence we see that the torsion-free direct summands of the pure subgroups *P*(*G*, *a*) need concern. Obviously these are all isomorphic to subgroups of **Q**.

Examples. **1.** $G = \mathbf{Z}(p) \oplus \mathbf{Q} \oplus \mathbf{Q} \in \mathcal{A} = \mathcal{S} \cap \mathcal{C}_1$.

Indeed, for every $a \in \mathbf{Q}$ we have (according to Proposition **4.1**) $P(G,a) = \mathbf{Z}(p) \oplus \mathbf{Q}$ and hence $G = P(G,a) \oplus \mathbf{Q} \in \mathcal{C}_1$. Obviously $G \in \mathcal{S}$.

2. Let *I* be an indecomposable torsion-free group of rank 2. Then $G = \mathbf{Z}_p \oplus I \in \mathcal{S} \setminus \mathcal{C}_1$.

Indeed, $G \in \mathcal{S}$ being clear, if we would have also $G \in C_1$ then (with the above notations - see $\mathcal{A}$) $I \cong G/\mathbf{Z}(p) \cong (P(G,a) \cap I) \oplus F$ contradicting the indecomposability of I.

3. For $G = \prod_{p \in \mathbf{P}} \mathbf{Z}(p)$ and $a = (\bar{1}, \bar{1}, ..., \bar{1}, ...)$ let $H = P(G,a) \oplus \mathbf{Q}$. We have $H \in C_1 \setminus \mathcal{S}$.

Indeed, $P(G,a) \notin \mathcal{S}$ (see [4]) implies $H \notin \mathcal{S}$ (from Consequence **4.4**) and $P(H, a) = P(G, a)$ so that $H \in C_1$.

4. *Conjecture*: let U be a nonsplitting mixed group such that $U \notin \mathcal{M}_3$ and I as above (indecomposable torsion-free of rank 2). Then $U \oplus I \in \mathcal{M}_3 \setminus (\mathcal{S} \cup C_1)$.

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