On torsion-free periodic rings

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Abstract

In this paper we characterize several large classes of periodic rings: periodic rings with identity, finite rank torsion-free periodic rings and rank 2 torsion-free periodic rings.

1 Introduction

There is a great deal of literature on periodic rings, respectively, torsion-free rings (especially of rank two). The aim of this paper is to provide a link between these two topics.

All groups considered here are Abelian, with addition the group operation. By order of an element we always mean the additive order of this element. All rings are associative but not necessarily with identity. The additive group of the ring R will be denoted by R^+. $\mathcal{M}_n(R)$ denotes the ring of all the $n \times n$ matrices with entries in R.

A ring R is called *periodic* if for each $x \in R$ the set $\{x, x^2, x^3, ...\}$ is finite, or equivalently, for each $x \in R$ there are positive integers $m(x)$, $n(x)$ such that $x^{m(x)} = x^{m(x)+n(x)}$. However, periodic rings can also be defined (see [23]) by requiring: (i) the multiplicative semigroup of R is periodic, or, (ii) if $a \in R$ then a power of a generates a finite subring. Examples of periodic rings are finite rings, nil rings and direct sums of matrix rings over finite fields. **Z**, the ring of all the integers, is not periodic.

Research on periodic rings (the term 'periodic' seems to have been first used by Chacron - see [18]) was mainly done in two directions:

- finding sufficient conditions on periodic rings which imply commutativity, Howard E. Bell being the prominent name in this direction (all over the last 40 years; e.g. see [12], [13], [14]) but also Hazar Abu-Khazam and Adil Yaqub (see [1], [2], [15] and [29]), respectively,

- finding structure results for some special classes of periodic rings (e.g., see [5], [6] and [14]).

However, it should be noticed that the starting point for these investigations was the Jacobson's theorem, whose proof contains many ideas which could be used also in more general contexts.

For later convenience we state here some **elementary properties** for a periodic ring:
(iii) Any infinite order element is a zero divisor (in the subring generated by itself).
(iv) Every idempotent in R has finite order.
(v) For each $a \in R$ some power of a is idempotent.

On the other hand, research on the additive groups of rings begun much more earlier. Defining ring structures on abelian groups was first done by Ross Beaumont (see [7]) who considered rings on direct sums of cyclic groups. Nearly at the same time, Tibor Szele investigated nil rings ([27]) and Beaumont and Zuckerman described the rings on subgroups of the rationals.

Satisfactory results were obtained later by Beaumont and Pierce for finite (and especially 2) rank torsion-free groups - see [9], [10]. Szele begun the program of investigating the additive structures of rings by the study of nilpotent rings (see [28]). However, a complete status of the results (previous to 1973) is given in Fuchs's treatise (see [22]). As of special interest for our paper, we also mention Freedman (see [20]) and Stratton - [26], who proved that non-nil torsion-free Abelian groups of rank two possess a unique minimal type, and their typeset has cardinality at most three. Here typeset(R), the typeset of R (or R^+), denotes the set of all types of the elements in R. For the definition of height and type of an element, we refer to [22]. For any group G and any type τ, $G(\tau) = \{x \in G | t(x) \geq \tau\}$. For a torsion-free group G, $E(G)$ denotes the endomorphism ring and $QE(G) = \mathbf{Q} \otimes_{\mathbf{Z}} E(G)$ the quasi-endomorphism ring.

Our main results can be summarized as follows.

In the first section we determine the structure of the periodic rings with identity.

In the second section we characterize periodic rings which have a finite rank torsion-free underlying additive group, obtaining as a by-product a special case confirmation of Kothe's conjecture.

In the last section we characterize the periodic torsion-free rings of rank two.

2 Periodic rings with identity

Given any ring R, for any fixed $a \in R$, the left and right multiplications with a are endomorphisms of R^+. Therefore, *fully invariant subgroups of R^+ are necessarily ideals in R*, no matter how multiplication is defined.

As a special case, *the torsion part $T(R)$ is a (two-sided) ideal of R*. Moreover, the primary components, R_p (p prime numbers) of R^+ are also ideals of R and every ring with torsion additive group decomposes (as a ring) $R = \bigoplus_{p \in \mathbf{P}} R_p$, $\mathbf{P}$ denoting the set of all prime numbers. A ring will be called a *p-ring* (p prime number) if its additive group is an (abelian) p-group. An abelian group is *bounded* if there exists a positive integer n such that $nR = \{0\}$.

Definition. A ring property Λ is called *non-Z*, if the ring of integers, does not have property Λ.

Examples of such properties are $\Lambda \equiv$ has zero divisors, or, $\Lambda \equiv$ periodic.

Proposition 2.1 *Let R be a ring with identity which satisfies a non-Z property Λ together with its subrings. Then R^+ is torsion. Moreover, R^+ is bounded.*

Proof. If 1_R denotes the identity, there is a canonical ring homomorphism $f : \mathbf{Z} \to R$ such that $f(n) = n1_R$, $\ker f = (\mathrm{char}(R))$, the ideal generated by the characteristics of R, and $\mathrm{im} f = \langle 1 \rangle \simeq \mathbf{Z}/\ker f$, the subring generated by 1_R. Together with R, $\langle 1 \rangle \simeq \mathbf{Z}/\ker f$ has property Λ and so, $\ker f = (\mathrm{char}(R)) \neq \{0\}$. Since $\mathrm{char}(R) = \mathrm{ord}_{R^+}(1_R)$, it follows that $1_R \in T(R)$. Hence $T(R) = R$, the torsion part being an ideal in R.

As for the last claim, if $n = \mathrm{char}(R) = \mathrm{ord}_{R^+}(1_R)$, for an arbitrary element $r \in R$, $nr = n(1_R r) = (n1_R)r = 0$, and $nR = \{0\}$. $\Box$

Corollary 2.1 ([22]) *A structure of ring with (left) identity exists on a torsion group G if and only if G is bounded.* $\Box$

Corollary 2.2 *Every periodic ring with identity is torsion (as a group). Moreover, it is bounded, and so a direct sum of cyclic groups.* $\Box$

As a special case, any semisimple periodic ring R is bounded (this will be used in the next section).

Corollary 2.3 *Every periodic ring with identity decomposes (as a ring) in a direct sum of p-rings. Each periodic p-ring is (as a group) a direct sum of cyclic p-groups.* $\Box$

Corollary 2.4 (see [24]) *A periodic ring with identity such that R^+ is finitely generated, is finite.* $\Box$

According to Corollary 2.3, the structure of periodic rings with identity reduces to p-rings which (as groups) are direct sums of cyclic p-groups. A special case of an early result due to Lászlo Fuchs settles this:

Theorem 2.1 (see [22]) *A multiplication μ on a direct sum $G = \bigoplus_{i \in I} \langle a_i \rangle$ of cyclic p-groups is completely determined by the values $\mu(a_i, a_j)$ with a_i, a_j running over this p-basis of G. Moreover, any choice of $\mu(a_i, a_j) \in G$ with a_i, a_j from this p-basis of G - subject to the condition $\mathrm{ord}(\mu(a_i, a_j)) \leq \min(\mathrm{ord}(a_i), \mathrm{ord}(a_j))$ - extends to a multiplication on G. The multiplication is associative [commutative] if (and only if) it is associative [commutative] on the p-basis $\{a_i\}_{i \in I}$.* $\Box$

More can be done (this is the last needed step): G being bounded, any element $a_{i_0}$of maximum order of this p-basis can be taken as identity of a ring, by letting a_{i_0} act as multiplication by 1 on $\langle a_{i_0} \rangle$ and by trivial multiplication on the other summands (see **120.8**, [22]).

It should be noted that a function $\mu : G\times G \to G$ is called a *multiplication on G* if it satisfies

$$\mu(a, b+c) = \mu(a,b) + \mu(a,c)$$
$$\mu(a+b, c) = \mu(a,c) + \mu(b,c)$$

for all a, b, c in G. Further, if $G = \bigoplus_{i\in I} H_i$ and H_i are fully invariant subgroups of G, multiplications on H_i $(i \in I)$ extend to multiplications on G (and conversely).

According to [22], an abelian group is called a *nil group* if there is no ring structure on G other than the zero-ring.

Theorem 2.2 (Szele [27]) *A torsion group is nil if and only if it is divisible.* □

3 Torsion-free periodic rings of finite rank.

Notice that for an arbitrary ring (denoting by $J(R)$ and $Nil(R)$ the Jacobson and the nil radicals, respectively) the following statements (known as *Köthe's conjecture*) are equivalent:

- The upper nilradical contains every nil left ideal;
- The sum of two nil left ideals is necessarily nil;
- $Nil(\mathcal{M}_n(R)) = \mathcal{M}_n(Nil(R))$ for all rings and for all n;
- $J(R[\lambda]) = Nil(R)[\lambda]$ for all rings R, where λ is a indeterminate commuting with all elements of ring.

From the elementary properties we mentioned in the Introduction it follows that any periodic torsion-free ring is nilpotent. Moreover (for an elementary proof see [24])

Lemma 3.1 *A torsion-free ring is periodic if and only if it is nil.*

Corollary 3.1 *If Kothe's conjecture holds, the matrix ring of a periodic torsion-free ring is also periodic.*

Next, recall that if R is a torsion-free ring of finite rank then $\mathbf{Q}R = \mathbf{Q} \otimes R$ becomes in a natural way a finite dimensional $\mathbf{Q}$-algebra (this comes back to Cartan and Eilenberg - see [16] or **119**, [22]). This is a divisible envelope for R^+ and the dimension of $\mathbf{Q}R$ over $\mathbf{Q}$ equals the rank of R^+. $\mathbf{Q}R$ may have an identity even if R does not (actually this happens exactly when there is an element e and an integer n such that $ex = nex$ for all elements x in R). Using the previous Lemma it follows that R is a periodic ring if and only if $\mathbf{Q}R$ is periodic.

The following result shows that in the torsion-free finite rank case, any periodic ring must be nilpotent (the converse obviously also holds).

Theorem 3.1 *Let R be a periodic torsion-free ring of rank n. Then $R^{n+1} = 0$.*

Proof. Since R is periodic, every element of R is nilpotent. Thus, the endomorphisms of the group R^+ of the form $t_r : R \to R$, $t_r(x) = rx$, are nilpotent endomorphisms, hence they belong to $N(\mathrm{E}(R^+))$, the nil-radical of the endomorphism ring if R^+. But (see [4, Theorem 9.1]) this nil-radical is nilpotent and so there exists a positive integer $k > 0$ such that $t_{r_1} \dots t_{r_k} = 0$ for any $r_1, \dots, r_k \in R$. Therefore R is a nilpotent ring.

Next, if R is a torsion-free ring of finite rank, the finite dimensional $\mathbf{Q}$-algebra $\mathbf{Q}R = \mathbf{Q} \otimes R$ is an artinian $\mathbf{Q}$-algebra. As previously noticed, R is a periodic ring if and only if $\mathbf{Q}R$ is periodic (indeed, $\forall s \in R, \exists m : r^m = 0$ implies $\forall \alpha s \in \mathbf{Q}R$ $(\alpha \in \mathbf{Q}) \exists m : (\alpha r)^m = \alpha^m r^m = 0$).

But $\mathbf{Q}R$ is an n-dimensional $\mathbf{Q}$-algebra, hence every strictly descending chain of $\mathbf{Q}$-ideals of $\mathbf{Q}R$ has at most n non-zero terms. Since $\mathbf{Q}R$ is nilpotent as a periodic ring, we use the chain $(\mathbf{Q}R) \geq (\mathbf{Q}R)^2 \geq \cdots \geq (\mathbf{Q}R)^{n+1}$, and the fact that if $(\mathbf{Q}R)^s = (\mathbf{Q}R)^{s+1}$ then $(\mathbf{Q}R)^s = (\mathbf{Q}R)^k$ for all $k > s$, to obtain $0 = (\mathbf{Q}R)^{n+1} = \mathbf{Q}R^{n+1}$. $\square$

Corollary 3.2 *Let R be a torsion-free ring of finite rank. Then R is periodic if and only if R is nilpotent.*

In the literature, rings which are finitely generated as rings, have been rarely studied. Obviously, if a ring is finitely generated as a group, it is also finitely generated as a ring.

Corollary 3.3 *Let R be a periodic ring of finite torsion-free rank. Then it is finitely generated as a ring if and only if it is finitely generated as a group.*

Proof. Let n be the rank of R^+. If $R = \langle r_1, \dots, r_m \rangle$, then

$$R^+ = \left\langle \prod_{i=1}^{k} x_i \mid k = 1, \dots, n,\ x_i \in \{r_1, \dots, r_m\} \right\rangle . \square$$

Remark. Actually more can be proved (see [24]): *if R is a commutative periodic ring, the two ways of being finitely generated are equivalent.*

Corollary 3.4 *If R is a periodic finite rank torsion-free ring then $\mathcal{M}_n(R)$ is periodic.* $\square$

Corollary 3.5 *If R is a rank 1 torsion-free periodic ring then $R^2 = 0$.* $\square$

4 Rank two torsion-free periodic rings

In this section R denotes a torsion-free ring of rank 2. We continue the discussion initiated by Beaumont and Wisner in [8] and continued by Beaumont and Pierce in [9].

The structure of rank 2 torsion-free groups which admit a non-trivial non-commutative multiplication was intensively investigated in [8] and [19]. First we show that such periodic rings are commutative. We recall that there exists (up to an isomorphism) only one structure of 2-dimensional nilpotent **Q**-algebra (see [17] or [9, Section 9]) and this is commutative. Using this and the **Q**-algebra **Q**R we obtain

Proposition 4.1 *A rank 2 torsion-free periodic ring is commutative.* $\square$

An important result towards finding the structure of the not nil rank two torsion-free rings was a theorem due to Freedman and Stratton, [20] and [26]: *the typeset of a not nil rank two torsion-free ring possesses a unique minimal element, and has at most three elements.*

The next result determines rank 2 torsion-free groups which admit a non-trivial multiplication of periodic type.

Theorem 4.1 *Let G be a rank 2 torsion-free group. G admits a nontrivial multiplication of periodic type if and only if there exists a proper pure subgroup H of G such that* $(\text{type}(G/H))^2 \leq \text{type}(H)$.

Proof. Let R be a periodic ring such that $R^2 \neq 0$ and the additive group R^+ is isomorphic to G. Then there exists $r \in R$ such that the (left) multiplication with r (i.e., $t_r : R \to R$, $t_r(x) = rx$) is a non-zero endomorphism of R^+. Since $R^3 = 0$ we obtain $t_r^2 = 0$ so that $H = \ker(t_r)$ is a pure subgroup of R with $t_r(R) \leq H$. Therefore there is a non-zero monomorphism $R/H \to H$. Moreover, since H is of rank 1, for every $h \in H$ there is a rational number q and $x \in R$ such that $h = qrx$. Consequently $RH = 0$ and it follows that $r \notin H$. If x_1, $x_2 \in R$ there are integers m_i, n_i and elements $h_i \in H$ ($i \in \{1, 2\}$) such that $n_i x_i = m_i r + h_i$. Hence $n_1 n_2 x_1 x_2 = m_1 m_2 r^2 \in H$ and so $r^2 \neq 0$ and $(\text{type}(G/H))^2 \leq \text{type}(H)$.

Conversely, suppose H is a proper subgroup of G with $(\text{type}(G/H))^2 \leq \text{type}(H)$. Let $S \leq T$ be rational groups such that $1 \in S$ with $\text{type}(G/H) = \text{type}(S)$ and $\text{type}(H) = \text{type}(T)$. From the type hypothesis we can suppose $S^2 = \{s_1 s_2 | s_1, s_2 \in S\} \subseteq T$. Fix $a \in G$ such that $S(a + H) = G/H$ and $h \in H$ with $Th = H$ and define a multiplication as follows: if $x_1, x_2 \in G$ and $n_i x_i = m_i a + h_i$ with $\frac{m_i}{n_i} \in S$ and $h_i \in H$ for all $i \in \{1, 2\}$ then $x_1 x_2 = \frac{m_1 m_2}{n_1 n_2} h$. It is easy to verify that this multiplication defines a periodic ring structure on G. $\square$

Remark. From the previous proof notice that if G is a rank n torsion-free group which admits a non-trivial periodic ring multiplication then G has a non-zero nilpotent endomorphism. Hence, in the $n = 2$ case, using Theorem 7.1 from [25], the quasi-endomorphism ring of G must be one of the following matrix rings:

- $\mathcal{M}_2(\mathbf{Q})$, or
- the ring of all 2×2 rational triangular matrices, or
- the ring of all 2×2 rational triangular matrices with equal diagonal entries.

We summarize from [4] (Section 3) what we need in the sequel. For a torsion-free group G of rank 2 we have the following possible situations:

- (a) the quasi-endomorphism ring of G is isomorphic to $\mathcal{M}_2(\mathbf{Q})$ if and only if $G = H \oplus K$ with $\mathrm{type}(H) = \mathrm{type}(K)$ (i.e., G is homogeneous completely decomposable).

- (b) the quasi-endomorphism ring of G is isomorphic to the ring of all 2×2 rational triangular matrices if and only if $G = H \oplus K$ with $\mathrm{type}(H) < \mathrm{type}(K)$.

- (c) the quasi-endomorphism ring of G is isomorphic to the ring of all 2×2 rational triangular matrices with equal diagonal entries if and only if G is strongly indecomposable, $|\mathrm{typeset}(G)| = 2$ and G has a nilpotent endomorphism.

Notice that in all these cases $\mathrm{typeset}(A) = \{\tau_1, \tau_2\}$ with $\tau_1 \leq \tau_2$.

Here a torsion-free group G is *strongly indecomposable* if whenever $0 \neq k \in \mathbf{Z}$ and $kG \subseteq H \oplus K \subseteq G$ then $H = 0$ or $K = 0$.

Theorem 4.2 *A rank 2 torsion-free group G admits a non-trivial periodic ring structure if and only if one of the following conditions holds:*
i) G is homogeneous completely decomposable of idempotent type, or
ii) $G = H \oplus K$ with $\mathrm{type}(H)^2 < \mathrm{type}(K)$, or
iii) G is strongly indecomposable, $\mathrm{typeset}(G) = \{\tau_1, \tau_2\}$ with $\tau_1 < \tau_2$ and $\mathrm{type}(G/G(\tau_2))^2 \leq \tau_2$.

Proof. The (i) case corresponds to (a) in the preceding discussion. In this situation every pure subgroup is a direct summand, hence the kernel $\ker(f)$, for every nilpotent endomorphism f of G, is a direct summand too. Then $\mathrm{type}(G/\ker(f)) = \mathrm{type}(\ker(f)) = \mathrm{type}(G)$ and so $\mathrm{type}(G)$ is idempotent. The same conclusion can be deduced from [26].

If G satisfies one of the conditions (b) or (c), the $\mathrm{typeset}(G) = \{\tau_1, \tau_2\}$ with $\tau_1 < \tau_2$. If f is a non-zero nilpotent endomorphism of G then $\ker(f) = G(\tau_2)$ (see [4], section 3). The proof is now complete using Theorem 4.1. □

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