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Trace 1, 2×2 matrices over principal ideal domains are exchange

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ABSTRACT. We prove that trace 1 matrices over principal ideal domains are exchange and characterize 2×2 exchange matrices over commutative domains. In addition, we emphasize large classes of not exchange 2×2 and 3×3 integral matrices.

1. INTODUCTION

An element a in a unital ring R is called *clean* ([8]) if $a = e + u$ with idempotent e and unit u and e-*clean*, if we intend to emphasize the idempotent. A clean element is called *trivial* if its decomposition uses a trivial idempotent i.e., 0 or 1. These are the 0-clean and 1-clean elements, respectively.

Also in [8], exchange elements (called *suitable*) were defined into four equivalent ways. One of these is: an element a in a ring R is (left) *exchange* if there is an idempotent e such that $e - a \in R(a - a^2)$. Every left exchange element is also right exchange and conversely. Clean elements are exchange ([8]).

Clean 2×2 integral matrices are characterized (see e.g. [2]) by pairs of one quadratic Diophantine equation (in two variables) and one linear Diophantine equation (in three variables). Since nowadays such equations are instantly solved by computer (see [1], [9] respectively [7]), it would be useful to have a criterion (checkable by computer) to decide on the exchange property of a matrix in $\mathcal{M}_2(\mathbb{Z})$.

In this note we characterize the exchange 2×2 matrices over *commutative domains*. It turns out that these are still characterized by pairs of conditions: but now one quadratic equation (in three variables) and one linear equation (in three or four variables).

Moreover, we prove that every trace 1, 2×2 matrix over a *principal ideal domain* is exchange.

In addition, we prove two special results, which mainly permit to emphasize large classes of not exchange 2×2 but also 3×3 *integral* matrices.

Whenever it is more convenient, we will use the widely accepted shorthand "iff" for "if and only if" in the text. All rings we consider are unital.

2. EXCHANGE 2×2 INTEGRAL MATRICES

First notice that an element a in a ring R is exchange iff there is an element $m \in R$ such that $c := a + m(a - a^2)$ is an idempotent. Observe that if this idempotent is trivial, then a is clean and so exchange. More precisely, we have the following

Lemma 2.1. *Let a be an element in a Dedekind finite ring R. There exists an element $m \in R$ such that*

(i) $a + m(a - a^2) = 0$ iff $1 - a$ is a unit (i.e. 1-clean);

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(ii) $a + m(a - a^2) = 1$ *iff* a *is a unit (i.e.* 0*-clean).*

Proof. The equality can be written as (i) $(1 - ma)(1 - a) = 1$ and (ii) $[1 + m(1 - a)]a = 1$, respectively, so since R is Dedekind finite, $1 - a$ is a unit, or a is a unit. Conversely, if $1 - a$ is a unit, we can chose $m = (1 - a)^{-1}$, and if a is a unit, we can chose $m = a^{-1}$. □

Remark 2.1. Notice that *tripotents*, i.e. elements $f \in R$ such that $f = f^3$ are (strongly) clean (see e.g. [3]) and so exchange.

In this section, R denotes a commutative domain. Notice that a 2×2 matrix E is idempotent iff $E \in \{0_2, I_2\}$, or else $\mathrm{Tr}(E) = 1$, $\det(E) = 0$. Also observe that, since left exchange elements are also right exchange and conversely, a matrix is exchange iff so is its transpose.

Let A be a 2×2 matrix over R. By Cayley-Hamilton Theorem,

$$A - A^2 = (1 - \mathrm{Tr}(A))A + \det(A)I_2.$$

If $\mathrm{Tr}(A) = 1$, and R is a principal ideal domain, we were able to prove the nice result (theorem) stated in the title.

For the proof we need the following

Lemma 2.2. *Let* a *and* c *be elements in a principal ideal domain* R*, with* $c \neq 0$*. The equation*

$$(ax - 1)[(a + 1)x - 1] = cy$$

has solutions $x, y \in R$*.*

Proof. Let $c = c_1c_2$ where c_1 is the product of the primes dividing c that do not divide a, and c_2 is the product of the primes dividing c that do divide a. Hence we know the following:

(i) $\gcd(c_1, c_2) = 1$;
(ii) $\gcd(c_1, a) = 1$;
(iii) $\gcd(c_2, a + 1) = 1$.

By Bézout's identity for (ii), there are elements x_1, z_1 such that $ax_1 + c_1z_1 = 1$ and so $ax_1 \equiv 1 \pmod{c_1}$. Similarly for (iii) there are elements x_2, z_2 such that $(a+1)x_2 + c_2z_2 = 1$ and so $(a + 1)x_2 \equiv 1 \pmod{c_2}$.

Owing to (i), by the Chinese Remainder Theorem there is an element x such that $x \equiv x_1 \pmod{c_1}$ and $x \equiv x_2 \pmod{c_2}$. Hence $ax \equiv 1 \pmod{c_1}$, $(a + 1)x \equiv 1 \pmod{c_2}$. We can rewrite this as $ax-1 = c_1y_1$, $(a+1)x-1 = c_2y_2$ where $y_1, y_2 \in R$. Let $y = y_1y_2$. Multiplying the two equations we obtain the claim in the statement. □

Theorem 2.1. *Every trace 1,* 2×2 *matrix over a principal ideal domain* R*, is exchange.*

Proof. As seen above, $A - A^2 = \det(A)I_2$. By definition, A is exchange iff there is a 2×2 matrix M such that $C := A + M(A - A^2) = A + \det(A)M$ is idempotent. We denote m_{ij}, $1 \leq i, j \leq 2$ the entries of M.

We first show that, for every trace 1 matrix $A = \begin{bmatrix} a+1 & b \\ c & -a \end{bmatrix}$ with $c \neq 0$, there exists a matrix M, with $m_{22} = -m_{11}$ and $m_{21} = 0$ such that C is a nontrivial idempotent, i.e., $\mathrm{Tr}(C) = 1$ and $\det(C) = 0$. In what follows, we can suppose $\det(A) \neq 0$. Otherwise, A is an idempotent, and so clean and exchange.

A simple computation gives $\mathrm{Tr}(C) = a + 1 + \det(A)m_{11} - a + \det(A)m_{22} = 1$ iff $\det(A)(m_{11} + m_{22}) = 0$ and $\det C = 0$ iff $(a + 1 + \det(A)m_{11})(-a + \det(A)m_{22}) - (b + \det(A)m_{12})(c + \det(A)m_{21}) = 0$.

$\mathrm{Tr}(C) = 1$ holds if $m_{22} = -m_{11}$, and if we replace $m_{22} = -m_{11}$ and $m_{21} = 0$ in the second equality we obtain (dividing by $\det(A)$)

$$1 - (2a+1)m_{11} - \det(A)m_{11}^2 - cm_{12} = 0.$$

Here $-\det(A) = a(a+1) + bc$ and we should show that for any $a, c \in R$, there is an m_{11} such that c divides $a(a+1)m_{11}^2 - (2a+1)m_{11} + 1 = (am_{11} - 1)[(a+1)m_{11} - 1]$. Equivalently, the quadratic Diophantine equation

$$(ax-1)[(a+1)x-1] - cy = 0$$

has solutions for any $a, c \in R$. Since $c \neq 0$, the existence follows from the previous lemma.

The remaining case, using our previous observation on the transpose of an exchange matrix (if $c = 0$ but $b \neq 0$), are the diagonal trace 1, 2×2 matrices, $A = \begin{bmatrix} a+1 & 0 \\ 0 & -a \end{bmatrix}$. Since for $a \in \{0, -1\}$ we obtain E_{11}, E_{22} which are idempotents, so clean and exchange, we may assume $a \notin \{0, -1\}$ (and so $\det(A) \neq 0$). We preserve $m_{22} = -m_{11}$ and so $\det(C) = 0$ reduces to

$$1 - (2a+1)m_{11} + a(a+1)(m_{11}^2 + m_{12}m_{21}) = 0.$$

Now we chose $m_{11} = 2a+1$ and $m_{12}m_{21} = -(2a-1)(2a+3)$, and the proof is complete. $\square$

Next, we prove a characterization for exchange matrices A with $\mathrm{Tr}(A) =: t \neq 1$ over any commutative domain. To simplify the writing we shall use the following notations: $\alpha = \det(A) + (1-t)(a+t)$, $\beta = (1-t)c$, $\gamma = (1-t)b$ and $\delta = \det(A) - (1-t)a$.

Theorem 2.2. *A trace $t \neq 1$, 2×2 matrix $A = \begin{bmatrix} a+t & b \\ c & -a \end{bmatrix}$ is exchange over a commutative domain R iff A is a unit, or $I_2 - A$ is a unit, or A is an idempotent, or*

(i) if $\beta = 0$, there exist $m_{11}, m_{12}, m_{21} \in R$ such that

$$(a+t+\alpha m_{11})(-a+1-t-\alpha m_{11}) = (b+\gamma m_{11}+\delta m_{12})\alpha m_{21}$$

and m_{22} with $1-t = \alpha m_{11} + \gamma m_{21} + \delta m_{22}$.

(ii) if $\delta = 0$, there exist $m_{11}, m_{21}, m_{22} \in R$ such that

$$(a+1-\gamma m_{21})(-a+\gamma m_{21}) = (b+\gamma m_{11})(c+\alpha m_{21}+\beta m_{22})$$

and m_{12} with $1-t = \alpha m_{11} + \beta m_{12} + \gamma m_{21}$.

(iii) if $\beta \neq 0 \neq \delta$, there exist $m_{11}, m_{12}, m_{21} \in R$ such that

$$\delta(a+t+\alpha m_{11}+\beta m_{12})[-a+1-t-(\alpha m_{11}+\beta m_{12})] =$$
$$(b+\gamma m_{11}+\delta m_{12})\{c\delta+\alpha\delta m_{21}+\beta[1-t-(\alpha m_{11}+\beta m_{12}+\gamma m_{21})]\}$$

and m_{22} with $1-t = \alpha m_{11} + \beta m_{12} + \gamma m_{21} + \delta m_{22}$.

Proof. As mentioned before $A - A^2 = (1-t)A + \det(A)I_2$ and we have only to discuss the case C is a nontrivial idempotent (otherwise, by Lemma 2.1, A is 0-clean or 1-clean).

Now $C =$
$\begin{bmatrix} a+t & b \\ c & -a \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \det(A)+(1-t)(a+t) & (1-t)b \\ (1-t)c & \det(A)-(1-t)a \end{bmatrix}$ and

$\mathrm{Tr}(C) = 1$ iff $(m_{11}+m_{22})\det(A) + (1-t)[m_{11}(a+t) + m_{12}c + m_{21}b - m_{22}a - 1] = 0$, and

$\det(C) = 0$ iff $[a+t+m_{11}\det(A) + (1-t)(m_{11}(a+t)+m_{12}c)][-a+m_{22}\det(A) + (1-t)(m_{21}b - m_{22}a)] =$

$= [b+m_{12}\det(A)+(1-t)(m_{11}b-m_{12}a)][c+m_{21}\det(A)+(1-t)(m_{21}(a+t)+m_{22}c)]$.

For the proof, we show that using $\mathrm{Tr}(C) = 1$ we can always eliminate m_{22} or m_{12} from the equality $\det(C) = 0$, this way obtaining a quadratic equation, to be solved in R.

Using the notations introduced above, $\mathrm{Tr}(C) = 1$ is equivalent to $\alpha m_{11}+\beta m_{12}+\gamma m_{21}+\delta m_{22} = 1-t$ and $\det(C) = 0$ is equivalent to

$$(a+t+\alpha m_{11}+\beta m_{12})(-a+\gamma m_{21}+\delta m_{22}) = (b+\gamma m_{11}+\delta m_{12})(c+\alpha m_{21}+\beta m_{22})(**)$$

We go into three cases.

(i) $\beta = 0$; since $t \neq 1$ this is equivalent to $c = 0$. Then we replace $\delta m_{22} = 1-t-(\alpha m_{11}+\gamma m_{21})$ into

$$(a+t+\alpha m_{11})(-a+\gamma m_{21}+\underline{\delta m_{22}}) = (b+\gamma m_{11}+\delta m_{12})\alpha m_{21}$$

and for any given m_{11}, this is a quadratic equation in unknowns m_{12}, m_{21}.

(ii) $\delta = 0$; this is equivalent to $\det(A) = (1-t)a$. Now we replace $\beta m_{12} = 1-t-(\alpha m_{11}+\gamma m_{21})$ into

$$(a+t+\alpha m_{11}+\underline{\beta m_{12}})(-a+\gamma m_{21}) = (b+\gamma m_{11})(c+\alpha m_{21}+\beta m_{22})$$

and for any given m_{11}, this is a quadratic equation in unknowns m_{21}, m_{22}.

(iii) $\beta \neq 0 \neq \delta$; this is equivalent to $c \neq 0$ and $\det(A) \neq (1-t)a$. In this case we multiply by β, $\alpha m_{11}+\beta m_{12}+\gamma m_{21}+\delta m_{22} = 1-t$ and by δ the equality $(**)$. Now we replace $\delta m_{22} = 1-t-(\alpha m_{11}+\beta m_{12}+\gamma m_{21})$ and $\beta\delta m_{22} = \beta[1-t-(\alpha m_{11}+\beta m_{12}+\gamma m_{21})]$ into

$$\delta(a+t+\alpha m_{11}+\beta m_{12})(-a+\gamma m_{21}+\underline{\delta m_{22}}) = (b+\gamma m_{11}+\delta m_{12})(c\delta+\alpha\delta m_{21}+\underline{\beta\delta m_{22}})$$

and for any given m_{11}, this is a quadratic equation in unknowns m_{12}, m_{21}.

The equations we obtain are displayed in the statement of the theorem. $\square$

Example 2.1. 1) $A = \begin{bmatrix} 3 & 3 \\ 0 & -1 \end{bmatrix}$. For $m_{11} = 0$ we have to solve the quadratic equation $(3-2m_{12})m_{21} = 1$. Among the solutions, if we chose $(m_{12}, m_{21}) = (2, -1)$ we get $M = \begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$, an idempotent. The linear equation becomes $3m_{21}+2m_{22} = 1$, also verified by $m_{21} = -1$, $m_{22} = 2$. So A *is exchange* over any ring.

Example 2.2. $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$. Here $\det(A) = (1-t)a$.

The linear equation is $m_{11}+3m_{21} = 1$ which has no solution for $m_{11} = 0$, unless 3 is a unit.

However, for $m_{11} = 1$ we have $m_{21} = 0$ and the linear equation is verified for any m_{12}, m_{22}. The quadratic equation is $36m_{21}^2 = 0$, also verified by $m_{21} = 0$. Hence we can chose $M = \begin{bmatrix} 1 & u \\ 0 & v \end{bmatrix}$ (arbitrary elements u, v) and $C = \begin{bmatrix} 0 & -3 \\ 0 & 1 \end{bmatrix}$ is idempotent. So A *is exchange* over any ring.

Example 2.3. $A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$. Here $c \neq 0$ and $\det(A) \neq (1-t)a$.

For $m_{11} = 0$ we have to solve the quadratic equation $3(1-m_{12})(-2+6m_{21}) = 0$ with solution $m_{12} = 1$ and arbitrary m_{21}, say $(m_{12}, m_{21}) = (1, y)$. Then $m_{22} = -y$ and for $M = \begin{bmatrix} 0 & 1 \\ y & -y \end{bmatrix}$ we obtain $C = \begin{bmatrix} 0 & 0 \\ 1-2y & 1 \end{bmatrix}$, an idempotent. The linear equation becomes $-m_{12}-3m_{21}-3m_{22} = 1$, also verified by $m_{12} = 1, m_{21} = y, m_{22} = -y$. So A *is exchange* over any ring.

Example 2.4. $A_k = \begin{bmatrix} 2k+1 & 0 \\ 0 & 0 \end{bmatrix}$ for any integer k. Here (i) amounts to

$$t(1-t)[1+(1-t)m_{11}](1-tm_{11}) = 0.$$

Since $m_{11} = \frac{1}{t-1} \in \mathbb{Z}$ only for $t \in \{0,2\}$ (which are not odd), and $m_{11} = \frac{1}{t} \in \mathbb{Z}$ only for $t \in \{\pm 1\}$, such matrices are exchange only if $t \in \{\pm 1\}$. Indeed, E_{11} is an idempotent, so exchange and $-E_{11}$ is exchange as tripotent. Hence all A_k, $k \notin \{-1,0\}$ *are not exchange* over $\mathbb{Z}$.

Remark 2.2. 1) When checking for exchange a 2×2 matrix A, one should first verify whether A or $A - I_2$ are invertible. If this fails then we use the previous (general) theorem. For instance, for the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ (which is a unit) (ii) is not (always) applicable (e.g. over $\mathbb{Z}$). Indeed, by computation $C = \begin{bmatrix} 1 & 3(1-m_{11}) \\ 0 & 1-3m_{21} \end{bmatrix}$ so $\mathrm{Tr}(C) = 2 - 3m_{21} \neq 1$ unless 3 is a unit. Actually, $C = C^2$ iff the left column of M is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (arbitrary right column) and so $C = I_2$.

2) Related to our research, from [6] we recall

Corollary 2.1. *Let$R = M_2(S)$ where S is any ring, and let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R$. If b or c is a unit, then a is exchange in R.*

It can be shown that if we replace b by any unit (or symmetrically c) in Theorem 2.2, the equations there, have solutions. In order not to lengthen this paper we only discuss the b is a unit case for $t \neq 1$, that is, using Theorem 2.2.

The linear equation is $\alpha m_{11} + \beta m_{12} + (1-t)bm_{21} + \delta m_{22} = 1 - t$ which obviously admits the solution $m_{11} = m_{12} = m_{22} = 0$ and $m_{21} = b^{-1}$. It is easy to check that this solution also verifies the quadratic equation and so for $M = b^{-1}E_{21}$, $C = A + b^{-1}E_{21}(A - A^2)$ is an idempotent. Hence A is (indeed) exchange over any ring.

3. Special characterizations

In this section, in order to simplify the exposition, we deal with *integral* matrices. The astute reader will easily realize how the results that follow may be generalized over principal ideal domains.

We show how one can exploit the situation $A - A^2 = \mathcal{M}_2(n\mathbb{Z})$, for a positive $n \geq 2$.

First we have the following

Proposition 3.1. *Let $n \in \mathbb{Z}$, $n \geq 2$ and let $A \in \begin{bmatrix} n\mathbb{Z}+1 & n\mathbb{Z} \\ n\mathbb{Z} & n\mathbb{Z}+1 \end{bmatrix} \subset \mathcal{M}_2(\mathbb{Z})$. Then A is exchange iff A is a unit iff* $\det(A) \in \{\pm 1\}$.

Proof. One way is clear. Conversely, suppose A is exchange. Notice that $A^2 \in \begin{bmatrix} n\mathbb{Z}+1 & n\mathbb{Z} \\ n\mathbb{Z} & n\mathbb{Z}+1 \end{bmatrix}$ and so $A - A^2 = nB$ for some $B \in \mathcal{M}_2(\mathbb{Z})$. Then $C = A + M(A - A^2) = A + nMB$ and so $\mathrm{Tr}(C) \in 2 + n\mathbb{Z}$.

If $n = 2$, C is idempotent only if it is trivial (i.e. the trace is 0 or 2). For $A \in \begin{bmatrix} 2\mathbb{Z}+1 & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z}+1 \end{bmatrix}$ we have $I_2 - A = 2S$, $M(I_2 - A) = 2T$, $I_2 + M(I_2 - A) \in \begin{bmatrix} 2\mathbb{Z}+1 & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z}+1 \end{bmatrix}$ and so $C \neq 0_2$ because $C = [I_2 + M(I_2 - A)]A \in \begin{bmatrix} 2\mathbb{Z}+1 & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z}+1 \end{bmatrix}$. In the remaining case, $C = [I_2 + M(I_2 - A)]A = I_2$, so A is a unit (we use Lemma 2.1).

If $n \geq 3$, again C is idempotent only if it is trivial. Now $C \neq 0_2$ follows from $C \in \begin{bmatrix} n\mathbb{Z}+1 & n\mathbb{Z} \\ n\mathbb{Z} & n\mathbb{Z}+1 \end{bmatrix}$, and so only $C = I_2$ remains. Using again Lemma 2.1, A must be a unit. □

Proposition 3.2. *For $n \in \mathbb{Z}$, $n \geq 2$, let $A \in \mathcal{M}_2(n\mathbb{Z})$. The following conditions are equivalent*
(i) A is exchange.
(ii) there is a matrix M such that $A + M(A - A^2) = 0_2$.
(iii) $A - I_2$ is a unit.
(iv) A is 1-clean.

Proof. (i) ⇒ (ii) Suppose A is exchange. By hypothesis, A, A^2, $A - A^2$, $M(A - A^2)$ and $C = A + M(A - A^2)$, all belong to $\mathcal{M}_2(n\mathbb{Z})$. Hence $C \neq I_2$ and since $\mathrm{Tr}(C)$ is multiple of n, that is $\neq 1$ (i.e. C is not nontrivial), the idempotent $C = 0_2$. (ii) ⇒ (iii) Use Lemma 2.1. (iii) ⇒ (iv) and (iv) ⇒ (i) are obvious. □

Corollary 3.2. *If the entries of a 2×2 integral matrix A are not (collectively) coprime then A is 1-clean (i.e. $A - I_2$ is a unit) or A is not exchange.*

Corollary 3.3. *Let A be an arbitrary 2×2 integral matrix and $m \in \mathbb{Z}$, $m \notin \{-1, 0, 1\}$. Then mA is 1-clean or not exchange.*

Example 3.5. $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ *is* I_2-clean so *exchange* but $\begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix}$, for $m \geq 3$ *are not exchange* (this improves Example 4, p. 6).

Example 3.6. $2I_2$ *is* (I_2-clean and so) *exchange* but mI_2, for $m \geq 3$ *are not exchange*.

Finally, we can now settle two special cases: the *diagonal* and the (upper) *triangular* 2×2 integral exchange matrices.

Proposition 3.3. *A diagonal integral matrix $A = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}$ is exchange iff $A = \pm I_2$ or E_{11}, $I_2 + E_{22}$, or else u, v are coprime and of different parity, and if $A = \begin{bmatrix} 2k+1 & 0 \\ 0 & 2l \end{bmatrix}$ then there exist integers m_{11}, m_{12}, m_{21} such that $(1 - 2km_{11})[1 - (2k+1)m_{11}] = -2l(2l-1)m_{12}m_{21}$ and $k(2k+1)m_{11} + l(2l-1)m_{22} = -(k+l)$. The case $A = \begin{bmatrix} 2l & 0 \\ 0 & 2k+1 \end{bmatrix}$ is recovered by conjugation with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.*

Proof. Computation shows that

$$C = A + M(A - A^2) = \begin{bmatrix} u[1 + (1-u)m_{11}] & v(1-v)m_{12} \\ u(1-u)m_{21} & v[1 + (1-v)m_{22}] \end{bmatrix}$$

and so $\mathrm{Tr}(C) = u[1 + (1-u)m_{11}] + v[1 + (1-v)m_{22}]$ and $\det(C) = uv[1 + (1-u)m_{11} + (1-v)m_{22} + (1-u)(1-v)\det(M)]$.

If both u, v are even, A is not exchange by Corollary 3.2.

If both u, v are odd, A is not exchange by Proposition 3.1, excepting the units $\pm I_2$. Since $\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}$ is conjugate to $\begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix}$ by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (the exchange property is invariant to conjugations), in the remaining case, using again Corollary 3.2, we may suppose $A = \begin{bmatrix} 2k+1 & 0 \\ 0 & 2l \end{bmatrix}$ has coprime entries, that is, $\gcd(2k+1, 2l) = 1$. In order to get a characterization we need Theorem 2.2.

Since $b = c = 0$, we also have $\beta = \gamma = 0$ and we can use case (i). With the corresponding notations, $u := a+t$, $v := -a$, $t = u+v$, $\alpha = u(1-u)$ and $\delta = v(1-v)$. Then the Diophantine equations become $1 - u - v = u(1-u)m_{11} + v(1-v)m_{22}$ and $u(1-u)[1 + (1-u)m_{11}](1 - um_{11}) = u(1-u)v(1-v)m_{12}m_{21}$.

Since the case we deal with has odd u and even v, we can discard $u = 0$. For $u = 1$ (the equality holds for any m_{11}, m_{12}, m_{21}) the linear equation amounts to $1 - v = \pm 1$, i.e. $v \in \{0, 2\}$. These are the matrices E_{11} and $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ (e.g. $M = I_2 + E_{22}$).

So finally the problem is reduced to $[1 + (1-u)m_{11}](1 - um_{11}) = v(1-v)m_{12}m_{21}$ and, with the above notations $(1-2km_{11})[1-(2k+1)m_{11}] = -2l(2l-1)m_{12}m_{21}$, together with $k(2k+1)m_{11} + l(2l-1)m_{22} = -(k+l)$. □

Example 3.7. $\begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix}$ *is not exchange*. Here the linear equation $21m_{11} + 6m_{22} = -5$ has no solution, because $\gcd(21; 6) = 3$ is not a divisor of 5.

Example 3.8. $\begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$ *is exchange*. Now the linear equation is $21m_{11} - m_{22} = -4$, which has solutions. For $m_{11} = 1$, we get $30 = -2m_{12}m_{21}$ or $m_{12}m_{21} = -15$. Since $\delta m_{22} = 1-t-\alpha m_{11}$ we obtain $m_{22} = -17$. For example, if $M = \begin{bmatrix} 1 & -3 \\ 5 & -17 \end{bmatrix}$ then $C = \begin{bmatrix} -35 & 6 \\ -210 & 36 \end{bmatrix}$, an idempotent.

Example 3.9. The matrices $A_l = \begin{bmatrix} 4l-1 & 0 \\ 0 & 2l \end{bmatrix}$, $l \notin \{0, 1\}$ *are not exchange*.

The linear equation $k(2k+1)m_{11} + l(2l-1)m_{22} = -(k+l)$ becomes $(2l-1)(4l-1)m_{11} + l(2l-1)m_{22} = -(3l-1)$. Since $\gcd(4l-1; l) = 1$ it follows that $\gcd((2l-1)(4l-1); (2l-1)l) = 2l-1$ and this (linear Diophantine) equation has solutions iff $2l-1$ divides $3l-1$. It is easy to see that this happens only for $l \in \{0, 1\}$.

For $l = 0$, $A_0 = -E_{11}$ *is* a tripotent, so *exchange*, and for $l = 1$, $A_1 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. For $M = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ we get $C = \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix}$, an idempotent. So A_1 *is exchange*.

We just mention that among matrices of form $\begin{bmatrix} 7 & 0 \\ 0 & s \end{bmatrix}$, these *are exchange* for $s \in \{-4, 2, 6, 8, 12, 24\}$ but *not exchange* for $s \in \{-2, 0, 4, 10, 16, 18, 20, 22\}$.

Proposition 3.4. *An upper triangular integral matrix* $A = \begin{bmatrix} a+t & b \\ 0 & -a \end{bmatrix}$*, with* $b \neq 0$*,* $t \neq 1$*, which is not* 1*-clean, is exchange iff* $a, b, a+t$ *are (collectively) coprime, and there exist integers* $m_{11}, m_{12}, m_{21}, m_{22}$ *such that, with the notations introduced before Theorem 2.2,* $1 - t = \alpha m_{11} + \gamma m_{21} + \delta m_{22}$ *and*

$$(a + t + \alpha m_{11})(-a + 1 - t - \alpha m_{11}) = (b + \gamma m_{11} + \delta m_{12})\alpha m_{21}.$$

Proof. This is Corollary 3.2, and Theorem 2.2, (i). □

Example 3.10. $\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$. The entries are not coprime, but the matrix is I_2-clean, so *exchange* over $\mathbb{Z}$.

Example 3.11. $\begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}$. The entries are not coprime, the matrix is not I_2-clean, so it is *not exchange* over $\mathbb{Z}$.

Example 3.12. $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$. The linear equation amounts to $2m_{11} + 9m_{21} + 2m_{22} = 3$ with solutions $m_{11} = 6 + 9u - v$, $m_{21} = -1 - 2u$, $m_{22} = v$ (arbitrary elements u, v).

The quadratic equation is $(2 - 2m_{11})(-1 + 2m_{11}) = 2(3 - 9m_{11} - 2m_{12})m_{21}$. For $u = -1$, $v = -2$ we get $m_{11} = -1$, $m_{21} = 1$, $m_{22} = -2$ and then $m_{12} = 3$. Indeed, for $M = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$ we get $C = \begin{bmatrix} 4 & 6 \\ -2 & -3 \end{bmatrix}$, an idempotent. Hence the matrix *is exchange* over any ring.

For $M = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$, we obtain the trivial idempotent $C = 0_2$ (not covered by the equations above).

4. Application

In [6] we can find the following

Theorem 5.12 *Let e be an idempotent in a ring R and $a = b + \varepsilon$ with $b \in S := eRe$, $\varepsilon \in \mathrm{Idem}(\overline{e}R\overline{e})$. Then a is exchange in R iff b is exchange in eRe* (here $\overline{e} = 1 - e$, the complementary idempotent).

There are two special cases which are related to our research.

1) $R = \mathcal{M}_3(\mathbb{Z})$ with $e = E_{11} + E_{22}$ and $\overline{e} = E_{33}$. In this case we identify $S = eRe$ with $\mathcal{M}_2(\mathbb{Z})$ and $\overline{e}R\overline{e}$ with $\mathbb{Z}$.

2) $R = \mathcal{M}_3(\mathbb{Z})$ with $e = E_{11}$ and then $\overline{e} = E_{22} + E_{33}$. In this case we identify $S = eRe$ with $\mathbb{Z}$ and $\overline{e}R\overline{e}$ with $\mathcal{M}_2(\mathbb{Z})$.

Using this, we obtain two consequences. We use block representations of 3×3 matrices ($\mathbf{0}$ denotes a 2-row, or a 2-column).

Corollary 4.4. *Let $U \in \mathcal{M}_2(\mathbb{Z})$ and $\varepsilon \in \{0, 1\} \subset \mathbb{Z}$. Then $A = \begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & \varepsilon \end{bmatrix}$ is exchange in $\mathcal{M}_3(\mathbb{Z})$ iff U is exchange in $\mathcal{M}_2(\mathbb{Z})$.*

Corollary 4.5. *Let $b \in \mathbb{Z}$ and $E = E^2 \in \mathcal{M}_2(\mathbb{Z})$. Then $A = \begin{bmatrix} b & \mathbf{0} \\ \mathbf{0} & E \end{bmatrix}$ is exchange in $\mathcal{M}_3(\mathbb{Z})$ iff b is exchange in $\mathbb{Z}$ iff $b \in \{-1, 0, 1, 2\} \subset \mathbb{Z}$.*

Therefore, using the results in the previous section we can generate plenty of *not exchange* 3×3 matrices.

Corollary 4.6. *The following 3×3 matrices are not exchange for any $n \in \mathbb{Z}$, $n \geq 2$:*

(a) $\begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & \varepsilon \end{bmatrix}$ *for* $U \in \begin{bmatrix} n\mathbb{Z}+1 & n\mathbb{Z} \\ n\mathbb{Z} & n\mathbb{Z}+1 \end{bmatrix}$, $\det(U) \notin \{\pm 1\}$ *and* $\varepsilon \in \{0, 1\}$,

(b) $\begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & \varepsilon \end{bmatrix}$ *for* $U \in \mathcal{M}_2(n\mathbb{Z})$, $\det(U - I_2) \notin \{\pm 1\}$ *and* $\varepsilon \in \{0, 1\}$,

(c) $\begin{bmatrix} b & \mathbf{0} \\ \mathbf{0} & E \end{bmatrix}$ *with any 2×2 idempotent E and $b \in \mathbb{Z} - \{-1, 0, 1, 2\}$.*

Final comments. 1) Among other things, Horia F. Pop wrote a program which, given a 2×2 matrix A, prints all the matrices M such that $B = A + M(A - A^2) = B^2$. In order to avoid a redundant search, the matrices M are searched incrementally with the nonnegative integer z starting at 0 and incremented by 1 for as long as it is deemed necessary. For each distinct value of z, only the matrices M with all elements in the closed interval $[-z, z]$ and having at least one element of absolute value equal to z are tested. This procedure has the advantage of splitting the set of all matrices M with integer elements into distinct subsets, covered one at a time. The program was decisive in proving some of our results.

2) In view of the result mentioned in the title, we could wonder whether

(a) trace 1, 2×2 matrices are (even) clean ?

(b) the result does hold for 3×3 matrices ?

Both questions have *negative answer*: $\begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}$ is not clean (see Theorem 4, [4]), and $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is not exchange (by Corollary 4.6, (a)).

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