The total number of subgroups of a finite Abelian group.

Grigore Călugăreanu *

Abstract. In this note, steps in order to write a formula that gives the total number of subgroups of a finite abelian group are made.

1 Introduction

It is well-known (Frobenius-Stickelberger, 1878, see [4]) that *a finite abelian group is the direct sum of a finite number of cyclic groups of prime power orders.* Obviously, their subgroups have the same structure.

However, when it comes to draw the subgroup lattice of a given finite abelian group, or to find out the **total** number of subgroups this group has, this can be a difficult task. It is the purpose of this note to describe a new method which partially solves these two problems.

First of all, let us mention some reductions which bring us closer to the real problem.

Let G be a finite abelian group and $|G| = n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$, the decomposition of its order into prime power factors. If $G = G_{p_1} \oplus G_{p_2} \oplus ... \oplus G_{p_k}$ is the corresponding primary decomposition, then, denoting by $L(G)$ the subgroup lattice of G, $L(G) \simeq L(G_{p_1}) \times L(G_{p_2}) \times ... \times L(G_{p_k})$, the direct product of the corresponding subgroup lattices (Suzuki [11]).

In the sequel we denote by $N(G)$ the number of subgroups of the group G. Hence $N(G) = \prod_{i=1}^{k} N(G_{p_s})$ and our counting problem is reduced to p-groups. Moreover, we shall consider that, given the subgroup lattices of the primary components, one can construct, the direct product of these lattices, and this finally gives the subgroup lattice of G. Therefore both problems reduce to primary groups.

In the above decomposition, the subgroup lattice $L(G)$ has generally more subgroups than the direct product of the subgroup lattices of the direct components.

As for our first problem, we will emphasize, which are the subgroups one has to add to the direct product of the subgroup lattices of the direct components. This will be possible using the key result described in the next section.

As for our second problem, we will find a formula which gives the total number of subgroups of an abelian finite group whose p-ranks do not exceed two.

Formulas which give the number of subgroups of type μ of a given finite p-group of type λ were given by Delsarte (see [2]), Djubjuk (see [3]), and Yeh (see [12]) in 1948. The reader may consult an excellent survey on this topic together with connections to symmetric functions written by L. Butler (see [1]). We mention below two of these formulas.

The application of the theory of symmetric functions to the study of abelian groups begins with P. Hall's unpublished work in the 1950's (according to Macdonald [7], these were conjectured in 1900 by Steinitz, see [10]). Hall proves that the number of subgroups H of type μ of a finite abelian p-group G of type λ such that the type of G/H is ν, is a

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polynomial in p with integer coefficients, the **Hall polynomial** $g^{\lambda}_{\mu\nu}$. Moreover, he finds its degree and leading coefficient and shows $g^{\lambda}_{\mu\nu} = g^{\lambda}_{\nu\mu}$.

Lemma 1.4.1([1]) *For any partitions $\mu \subset \lambda$, the number of subgroups of type μ in a finite abelian p-group of type λ is*

$$\alpha_\lambda(\mu; p) = \prod_{j\geq 1} p^{\mu'_{j+1}(\lambda'_j - \mu'_{j+1})} \begin{bmatrix} \lambda'_j - \mu'_{j+1} \\ \mu'_j - \mu'_{j+1} \end{bmatrix}_p$$

where λ' is the conjugate of λ, and μ' is the conjugate of μ.

Remarks. - If $\lambda = (\lambda_1, \lambda_2 ...)$ is a partition, then the conjugate partition $\lambda' = (\lambda'_1, \lambda'_2 ...)$ has components defined as follows: λ'_j is the number of parts λ'_i such that $\lambda'_i \geq j$. (So the jth row of the Ferrers diagram of λ' is the jth column of the Ferrers diagram of λ.)

- Here $\begin{bmatrix} n \\ k \end{bmatrix}_p = \dfrac{\prod_{i=1}^{n}(p^i - 1)}{\prod_{i=1}^{k}(p^i-1)\prod_{i=1}^{n-k}(p^i-1)}$ denote the Gaussian coefficients, the number of all the k-dimensional subspaces of an n-space.

- The formula given by Djubjuk (see [3]) is similar.

Here is a second formula (adapted from [2]):

$$N(X;Y) = \frac{p^{s_1r_2 + s_2r_3 + \dots s_{k-1}r_k} \prod_{i_1=r_2}^{r_1-1}(p^{s_1} - p^{i_1}) \prod_{i_2=r_3}^{r_2-1}(p^{s_2} - p^{i_2}) \dots \prod_{i_1=0}^{r_k-1}(p^{s_k} - p^{i_k})}{p^{r_1r_2 + r_2r_3 + \dots r_{k-1}r_k} \prod_{i_1=r_2}^{r_1-1}(p^{r_1} - p^{i_1}) \prod_{i_2=r_3}^{r_2-1}(p^{r_2} - p^{i_2}) \dots \prod_{i_1=0}^{r_k-1}(p^{r_k} - p^{i_k})}$$

where $s_1 \geq s_2 \geq ... \geq s_k \geq 1$ is the signature of a group of type Y and $r_1 \geq r_2 \geq ... \geq r_k \geq 1$ is the signature of a subgroup of type X ($r_j \leq s_j$, $1 \leq j \leq k$).

With our above notation the connection between the invariants which give the type $Y = (n_1, n_2, ..., n_k)$ and the signature $(s_1, s_2, ..., s_k)$ is given by the following relations:

$$s_1 - s_2 = n_1, s_2 - s_3 = n_2, ..., s_{k-1} - s_k = n_{k-1}, s_k = n_k$$

or

$$s_1 = n_1 + n_2 + ... + n_k, s_2 = n_2 + n_3 + ... + n_k, ..., s_{k-1} = n_{k-1} + n_k, s_k = n_k.$$

However, for a given p-group, to sum up the numbers given by the above formulas is another difficult task (e.g., notice that for $t < k$ the signature of $G = \mathbf{Z}_{p^t} \oplus \mathbf{Z}_{p^k}$ is $(2, 2, ..., 2, 1, 1, ..., 1)$ and one has to sum up all the subgroups considering all the possible sub-signatures!). That's why, our method (which seems to be original, and natural since it describes directly the corresponding subgroup lattices) gives a formula for *the total number* of subgroups of a (finite) abelian rank two p-group which, to the best of our knowledge, did not yet appear in print.

2 The key result Our key result is antique. It was discovered by Goursat as early as 1890 (see [6]): it is possible to construct the subgroup lattice of a direct sum out of the subgroup lattices of the summands, and all the isomorphisms between "sections" ! A brief presentation follows (for details see [9]).

(a2) *Let U be a subgroup of $G = H \oplus K$. Then there is natural isomorphism*

$$\frac{(U+K)\cap H}{U\cap H} \simeq \frac{(U+H)\cap K}{U\cap K}.$$

Conversely, let $G = H \oplus K$ be a group and $W_1 \leq U_H \leq H$, $W_2 \leq U_K \leq K$ subgroups of the direct summands. For every isomorphism $\delta : \frac{U_H}{W_1} \longrightarrow \frac{U_K}{W_2}$ there exists a subgroup $U \leq G$ such that $U_H = (U+K)\cap H$, $U_K = (U+H)\cap K$, $W_1 = U\cap H$ and $W_2 = U \cap K$, namely

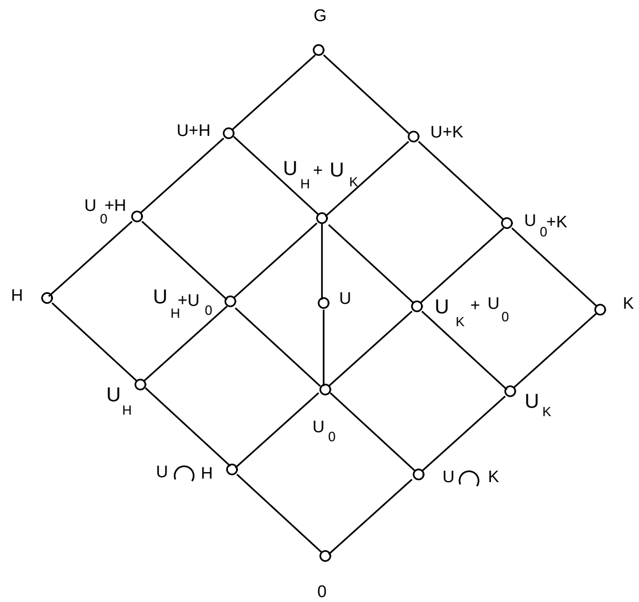
$$U = D(U_H, \delta) = \{x + y | x \in U_H, y \in \delta(x + W_1)\}.$$

Thus in order to recover the subgroups of a direct sum $H \oplus K$ we need the isomorphisms between the "sections" (i.e., intervals in the subgroup lattice) in $[0, H]$ respectively $[0, K]$.

Let $G = H \oplus K$. A subgroup D of G is called a *diagonal* in G (with respect to H and K) if $D + H = G = D + K$ and $D \cap H = 0 = D \cap K$.

If $\delta : H \longrightarrow K$ is an isomorphism then $D(\delta) = D(H, \delta) = \{x + \delta(x)|x \in H\} = (1 + \delta)(H)$ is a diagonal in G (with respect to H and K). Conversely, if D is a diagonal in G (with respect to H and K) there is a unique isomorphism $\delta : H \longrightarrow K$ such that $D = D(\delta)$.

More, there is a bijection between the diagonals (with respect to H and K) and isomorphisms of H and K. Every subgroup U of a direct sum $G = H \oplus K$ belongs to the direct product $\mathbf{L} = L(H) \times L(K)$ (i.e., has the form $H' \oplus K'$ for $H' \leq H$ and $K' \leq K$) or is a diagonal. The situation is best described by the following diagram



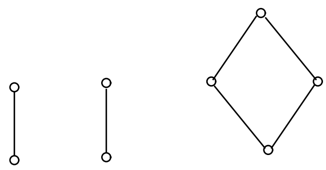
Using the projections from the direct sum, $U_H = p_H(U)$, $U_K = p_K(U)$, $U_0 = (U \cap H) \oplus (U \cap K) \in \mathbf{L}$ and $U_H \oplus U_K = (U + H) \cap (U + K) \in \mathbf{L}$. Moreover (see [5] where these are described as subdirect sums), $U_H \oplus U_K$ is *minimal* in $\mathbf{L}$ *including* U and $U_0 = (U \cap H) \oplus (U \cap K)$ is *maximal* in $\mathbf{L}$ *included in* U.

Finally, this allows us to reconstruct, beginning with the direct product of the subgroup lattices $L(H)$, $L(K)$ all the other subgroups, namely the diagonals, just looking at the isomorphisms between the "sections".

3 Examples

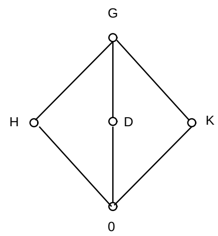
1) The Klein group $G = H \oplus K = \mathbf{Z}_2 \oplus \mathbf{Z}_2 = \{0, a, b, a+b | 2a = 0 = 2b\}$.

Each $L(\mathbf{Z}_2)$ is a chain with two elements and the direct product of these two chains is the four element lattice



Using the key result, we have to add as many diagonals as isomorphisms $\mathbf{Z}_2 \to \mathbf{Z}_2$ (i.e., $H \to K$). From $H = \{0, a\}$ to $K = \{0, b\}$ there is only one isomorphism so only one diagonal (namely $(1 + \delta)(H) = \{0, a + b\}$) has to be added to the direct product.

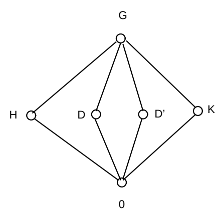
Hence G has the "diamond" as subgroup lattice



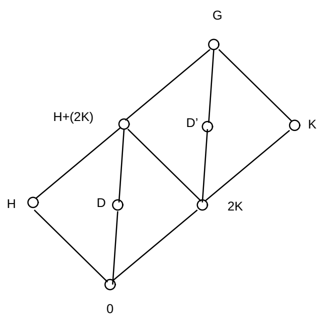
2) The elementary group $G = \mathbf{Z}_3 \oplus \mathbf{Z}_3 = \{0, a, 2a, b, 2b, a + b, a + 2b, 2a + b, 2a + 2b | 3a = 0 = 3b\}$.

The same two chains of two elements and four element direct product of lattices.

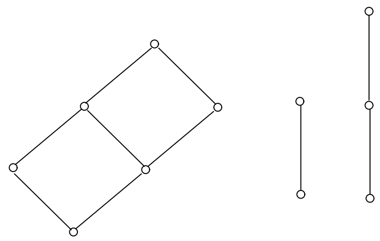
However, now there are two isomorphisms $\mathbf{Z}_3 \to \mathbf{Z}_3$, namely $\delta : \begin{pmatrix} 0 & a & 2a \\ 0 & b & 2b \end{pmatrix}$ and $\delta' : \begin{pmatrix} 0 & a & 2a \\ 0 & 2b & b \end{pmatrix}$. Hence we must add two diagonals $D = (1+\delta)(H) = \{0, a+b, 2a+2b\}$ respectively $D' = (1 + \delta')(H) = \{0, a + 2b, 2a + b\}$. The following diagram describes this



3) The group $G = \mathbf{Z}_2 \oplus \mathbf{Z}_4 = \{0, a, b, a+b, 2b, a+2b, 3b, a+3b | 2a = 0 = 4b\}$ has the subgroup lattice

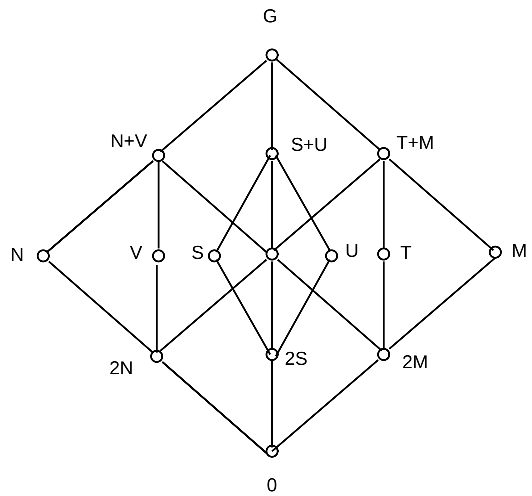


Indeed, to the direct product of chains



we have to add only two diagonals, D corresponding to the isomorphism $[0, H] \to [0, 2K]$ respectively D' to the isomorphism $[0, H] \to [2K, K]$.

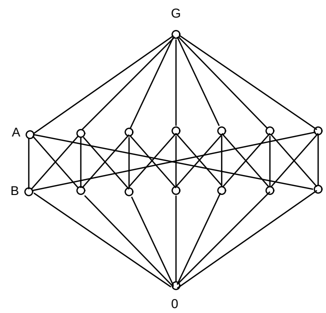
4) The group $G = \mathbf{Z}_4 \oplus \mathbf{Z}_4 = \langle a, b | 4a = 4b = 0 \rangle$ with cyclic subgroups $N = \langle a \rangle = \langle 3a \rangle$, $M = \langle b \rangle = \langle 3b \rangle$ has the subgroup lattice



Once again, we start by two 3-element chains (i.e., of length 2), and the direct product has 9 elements. For the first time we have here two kind of isomorphisms: four isomorphisms $\mathbf{Z}_2 \to \mathbf{Z}_2$ $([0, 2N] \to [0, 2M]$, $[0, 2N] \to [2M, M]$, $[2N, N] \to [0, 2M]$, $[2N, N] \to [2M, M])$

respectively two isomorphisms $\mathbf{Z}_4 \to \mathbf{Z}_4$ ($[0, N] \to [0, M]$ - $\mathbf{Z}_4$ has two automorphisms), so we add 6 diagonals, $2S$, T, V, $S + U$ and S, U, for a total of 15 subgroups.

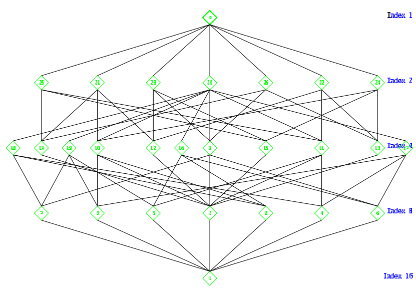
5) The elementary group $G = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ has the following subgroup lattice



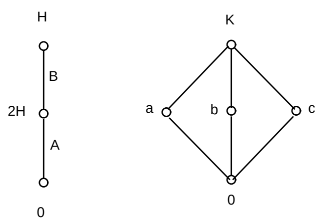
In order to recover this by using our key result, we consider $H = \mathbf{Z}_2$ with a 2-element chain subgroup lattice, and $K = \mathbf{Z}_2 \oplus \mathbf{Z}_2$, with the 'diamond' subgroup lattice (see Example 1).

We have $2 \times 5 = 10$ subgroups in the direct product + 6 diagonals to add, corresponding to the 6 isomorphisms associated to all the segments in the 'diamond'. That is, 16 subgroups indeed.

6) The 2-group $\mathbf{Z}_4 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$.



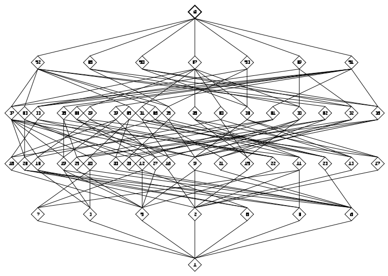
Now we take $H = \mathbf{Z}_4$ and $K = \mathbf{Z}_2 \oplus \mathbf{Z}_2$, so a 3-element chain and the 'diamond' as subgroup lattices, as follows



Hence we have $3 \times 5 = 15$ subgroups in the direct product, and we must add diagonals corresponding to 6 isomorphisms from $A = [0, 2H]$ to each of the **segments** on the 'diamond' and other 6 isomorphisms from $B = [2H, H]$ to the same 6 segments. That is $15 + 6 + 6 = 27$ subgroups.

Remark. Apparently there would be some isomorphisms from the interval (chain) $[0, H]$ to some 3-element chains in the 'diamond' (namely $[0, a, K]$, $[0, b, K]$ or $[0, c, K]$). Wrong! None of these actually is an interval in the subgroup lattice $L(K)$.

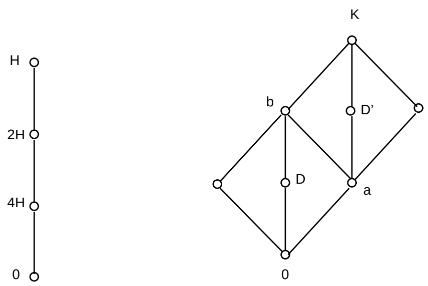
7) $G = \mathbf{Z}_2 \oplus (\mathbf{Z}_4 \oplus \mathbf{Z}_4)$.



First, the direct product of a 2-element chain and the lattice in Example 5. Thus $2 \times 15 = 30$ subgroups in the direct product + as many diagonals as isomorphisms $\mathbf{Z}_2 \to \mathbf{Z}_2$. Now we have as many such isomorphisms as many segments has the lattice $L(\mathbf{Z}_4 \oplus \mathbf{Z}_4)$; this number is 24. Finally we have $30 + 24 = 54$ subgroups.

8) $G = \mathbf{Z}_8 \oplus (\mathbf{Z}_2 \oplus \mathbf{Z}_4)$ Apparently similar with the previous, this Example emphasizes a new aspect.

First we have $4 \times 8 = 32$ subgroups in the direct product + 3×11 diagonals for the corresponding (simple) segments [hereafter also called 1-*segments*].



Next we must count 2-segments (on the right lattice there are no 3-segments which are intervals): 2 as (long) sides of the right rectangle **and** the chains (which actually are also intervals!) $[0, a, D']$ respectively $[D, b, K]$. Hence, $2 \times 2 \times 4 = 16$ more diagonals to be added (**2** for $[0, 2H]$ and $[4H, H]$, $|\mathrm{Aut}(\mathbf{Z}_4)| = 2(2-1) = \mathbf{2}$; see also next section, respectively the **4** 2-segments), for a total of $32 + 33 + 16 = 81$ subgroups.

9) $G = (\mathbf{Z}_2 \oplus \mathbf{Z}_2) \oplus (\mathbf{Z}_4 \oplus \mathbf{Z}_4)$ is the only group of rank four we discuss in this Section.

Consider $H = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and $K = \mathbf{Z}_4 \oplus \mathbf{Z}_4$ which have as subgroup lattices the 'diamond' (Example 1) and the lattice $L(\mathbf{Z}_4 \oplus \mathbf{Z}_4)$ from Example 5 which has 15 subgroups and 24 segments.

Hence we have $5 \times 15 = 75$ subgroups in the direct product, $6 \times 24 = 144$ diagonals corresponding to all the $\mathbf{Z}_2 \to \mathbf{Z}_2$ isomorphisms (between the 1-segments) and (for the first time), 5 isomorphisms of 'diamonds' (indeed, $L(\mathbf{Z}_4 \oplus \mathbf{Z}_4)$ has 5 intervals which are 'diamonds'). That is, $5 \times |\mathrm{Aut}(\mathbf{Z}_2 \oplus \mathbf{Z}_2)| = 5 \times 6 = 30$ (the Klein group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ has $3! = 6$ automorphisms).

So the total is $75 + 144 + 30 = 249$ subgroups.

4 The rank 2 formula

Diagonals, segments and automorphisms.

In what follows, 'segment' on the diagram that represents a given subgroup lattice has the usual geometric meaning and, as it is well-known, is drawn whenever a subgroup S is covered by another subgroup T (i.e., T/S is simple). Moreover, the term 'interval' (we have finally preferred it to the synonymous 'section') is used as follows: if S, T are subgroups of G, the interval $[S, T] = \{U \in L(G) | S \leq U \leq T\}$. As previously noted, this is generally not a chain (excepting cocyclic groups) nor a 'diamond'.

Notice that counting segments may be done by counting the segments in the direct product and adding 2 times the number of diagonals (obviously, each diagonal 'comes' with exactly two 1-segments - in producing the corresponding 'diamond'). For later purposes, the two 1-segments adjacent to a diagonal will be called *half-diagonals* (thus, diagonal means a subgroup, and half-diagonal means an 1-segment).

Recall (e.g. see [5]) that $|\mathrm{Aut}(\mathbf{Z}_n)| = \varphi(n)$ the Euler (arithmetic) function. Thus

$|\mathrm{Aut}(\mathbf{Z}_{p^t})| = \varphi(p^t) = p^t(1 - \frac{1}{p}) = p^{t-1}(p-1)$ gives the number of isomorphisms between two intervals of the form $[0, \mathbf{Z}_{p^t}]$ (i.e., chains of length t).

The formula.

In what follows we shall prove the formula giving the total number of subgroups for a (finite) rank two p-group.

Set $G = \mathbf{Z}_{p^t} \oplus \mathbf{Z}_{p^k}$ for arbitrary positive integers t and k.

First of all, we have the direct product of chains of length t respectively k, that is, $(t+1)(k+1)$ subgroups. Next, we have the diagonals corresponding to the automorphisms $\mathbf{Z}_p \to \mathbf{Z}_p$, which give $p-1$ diagonals for each pair of 1-segments, i.e., $tk(p-1)$.

Further, the diagonals corresponding to the automorphisms $\mathbf{Z}_{p^2} \to \mathbf{Z}_{p^2}$, which give $p(p-1)$ diagonals for each pair of (double) adjacent segments (2-segments). So $(t-1)(k-1)p(p-1)$ diagonals have to be added.

We must continue till we exhaust the adjacent $\min(t, k)$-length segments, which obviously is the chain $L(H)$ or the chain $L(K)$ (corresponding to $t \leq k$ respectively $k \leq t$). This chain produces $|k - t| + 1$ pairs of chains of length $\min(t, k)$, each giving $p^{\min(t,k)-1}(p-1)$ diagonals.

Therefore the total number of subgroups is

$$(t+1)(k+1) + tk(p-1) + (t-1)(k-1)p(p-1) + ...$$
$$... + 2(|k-t|+2)p^{\min(t;k)-2}(p-1) + (|k-t|+1)p^{\min(t;k)-1}(p-1).$$

In what follows a special sum will be used. For $t \le k$ denote by

$$S_{tk} = \sum_{i=0}^{t-1} (t-i)(k-i)p^i$$

Considering the polynomial $f = 1 + X + X^2 + ... + X^{t-1} \in \mathbf{Z}[X]$, this sum is

$$S_{tk} = tkf(p) - (t+k)pf'(p) + p[(Xf'(X))'](p)$$

(here $f'(X) = \dfrac{(t-1)X^t - tX^{t-1} + 1}{(X-1)^2}$ and
$(Xf'(X))' = \dfrac{(t-1)^2X^{t+1} - (2t^2 - 2t - 1)X^t + t^2X^{t-1} - X - 1}{(X-1)^3}$).

Hence S_{tk} is calculated as follows:

$$tk\frac{p^t - 1}{p-1} - (t+k)p\frac{(t-1)p^t - tp^{t-1} + 1}{(p-1)^2} + p\frac{(t-1)^2p^{t+1} - (2t^2 - 2t - 1)p^t + t^2p^{t-1} - p - 1}{(p-1)^3},$$

or

$$\frac{p^{t+2}(k-t+1) + p^{t+1}(t-k+1) - (t+1)(k+1)p^2 + (2tk + t + k - 1)p - tk}{(p-1)^3}.$$

Moreover, for two arbitrary positive integers t, k denote by $S_{\overline{tk}} = S_{\min(t;k),\max(t;k)}$.
Then we obtain

$$N(\mathbf{Z}_{p^t} \oplus \mathbf{Z}_{p^k}) = (t+1)(k+1) + (p-1)S_{\overline{tk}}$$

or finally

$$(t+1)(k+1) +$$
$$+\frac{p^{t+2}(|k-t|+1) + p^{t+1}(1-|k-t|) - (t+1)(k+1)p^2 + (2tk + t + k - 1)p - tk}{(p-1)^2}$$

$$= (t+1)(k+1) + \frac{p^{t+2}(|k-t|+1) + p^{t+1}(1-|k-t|) - p - [(t+1)p - t][(k+1)p - k]}{(p-1)^2}.$$

Remarks.- As for (finite) rank three p-groups, the use of the same 'key' result is possible but complicated (the previous presented Examples suggest which are the difficulties). It will be done elsewhere.

However, subgroups of a direct sum of three groups were already described in 1931 by Remak (see [8]). Actually this is a difficult paper to read (some 156 equations describe finally the corresponding 'key' result !).

The use of the corresponding isomorphism (for direct sum of two groups this was **(a2)** in our second Section) is more effective in giving the formula for the total number of subgroups for a rank three p-group:

(a3) *Let $G = H_1 \oplus H_2 \oplus H_3$ and U be a subgroup of G. There is natural isomorphism*

$$\frac{(L_1 \oplus L_2 \oplus L_3) + U}{L_1 \oplus L_2 \oplus L_3} \cong \frac{U}{N_1 + N_2 + N_3}$$

using the notations:

$B_1 = H_1 \cap (U + (H_2 \oplus H_3))$, $A_{12} = (U + H_1) \cap H_2$, $K_1 = (U + H_1) \cap (H_2 \oplus H_3)$, $C_1 = U \cap H_1$, $D_1 = U \cap (H_2 \oplus H_3)$, $L_1 = A_{12} \cap A_{13}$, $M_1 = A_{12} + A_{13}$, $N_1 = D_1 \cap (A_{12} \oplus A_{13})$, and the other combinations interchanging indexes 1,2,3.

One has to write the converse (as for the two summands case) and to establish the corresponding bijection between subgroups and isomorphism of intervals. This gives, in a similar way, the total number of subgroups.

5 Elementary abelian groups; recurrences.

Any finite elementary p-group, is represented as

$$\mathbf{Z}_p^n = \bigoplus_n \mathbf{Z}_p$$

for a positive integer n.

Denote by $N(\mathbf{Z}_p^n)$ its total number of subgroups and by n(1−seg($L(\mathbf{Z}_p^n)$)) the number of 1-segments the subgroup lattice $L(\mathbf{Z}_p^n)$ has. Using the Gaussian coefficients, one can define the *Galois numbers* $G_{n,p} = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_p$. Recall that for these numbers there is only a recursion (and no formula):

for all $N \in \mathbf{N}$ and p prime (power)

$$G_{0,p} = 1, \; G_{1,p} = 2,$$

$$G_{n+1,p} = 2G_{n,p} + (p^n - 1)G_{n-1,p}.$$

Since $\mathbf{Z}_p^n$ is an n -dimensional vector space over $\mathbf{Z}_p$, $\left[\begin{array}{c} n \\ k \end{array} \right]_p$ gives the number of subgroups of dimension (rank) k, respectively, $G_{n,p}$ gives the total number of subgroups, i.e.

$$N(\mathbf{Z}_p^n) = G_{n,p}.$$

Using our 'key result' , $\#\mathrm{Aut}(\mathbf{Z}_p) = p - 1$ and $\mathbf{Z}_p^n = \mathbf{Z}_p \oplus \mathbf{Z}_p^{n-1}$, that is $L(\mathbf{Z}_p)$ a chain with 2 elements, by counting the subgroups in the direct (lattices) product, respectively the diagonals, we obtain

$$N(\mathbf{Z}_p^n) = 2N(\mathbf{Z}_p^{n-1}) + (p-1)\mathrm{n}(1\mathrm{-seg}(L(\mathbf{Z}_p^{n-1}))).$$

Consequence.

$\mathrm{n}(1\mathrm{-seg}(L(\mathbf{Z}_p^n))) = \frac{1}{p-1}[N(\mathbf{Z}_p^{n+1}) - 2N(\mathbf{Z}_p^n)] = \frac{1}{p-1}[G_{n+1,p} - 2G_{n,p}] = \frac{p^n-1}{p-1}G_{n-1,p}.$

Hence

$$\mathrm{n}(1\mathrm{-seg}(L(\mathbf{Z}_p^n))) = (p^{n-1} + p^{n-2} + ... + p + 1)G_{n-1,p}.$$

Finally, we mention another straightforward recurrence one obtains by using the same 'key result' : each finite p-group G can be viewed as $\mathbf{Z}_{p^l} \oplus G'$ where G' is a (finite) direct sum of (finite) cocyclics, of order larger or equal to p^l. If we know how to count the 1-segments, 2-segments, ..., the l-segments in $L(G')$ obviously the total number of subgroups is obtained adding (hereafter $n(\mathrm{u}-\mathrm{seg})$ denotes the number of u-segments in G'):

$l(p-1)n(1-\mathrm{seg})$ diagonals corresponding to isomorphisms $\mathbf{Z}_p \to \mathbf{Z}_p$;
$+(l-1)p(p-1)n(2-\mathrm{seg})$ diagonals corresponding to isomorphisms $\mathbf{Z}_{p^2} \to \mathbf{Z}_{p^2}$;
$+(l-2)p^2(p-1)n(3-\mathrm{seg})$ diagonals corresponding to isomorphisms $\mathbf{Z}_{p^3} \to \mathbf{Z}_{p^3}$;
...
$+2p^{l-2}(p-1)n((\mathrm{l}-1)-\mathrm{seg})$ diagonals corresponding to isomorphisms $\mathbf{Z}_{p^{l-1}} \to \mathbf{Z}_{p^{l-1}}$;
$+p^{l-1}(p-1)n(\mathrm{l}-\mathrm{seg})$ diagonals corresponding to isomorphisms $\mathbf{Z}_{p^l} \to \mathbf{Z}_{p^l}$.

Hence

$$N(\mathbf{Z}_{p^l} \oplus G') = l(p-1)n(1-\mathrm{seg}) + (l-1)p(p-1)n(2-\mathrm{seg}) + (l-2)p^2(p-1)n(3-\mathrm{seg}) + \ldots$$
$$\ldots + 2p^{l-2}(p-1)n((\mathrm{l}-1)-\mathrm{seg}) + p^{l-1}(p-1)n(\mathrm{l}-\mathrm{seg}).$$

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