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SOME REMARKS ABOUT MUTUAL PSEUDOCOMPLEMENTS IN LATTICES

by

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1. Preliminaries. Let L be a modular lattice with 0 and 1. In what follows we shall deal with the notion of pseudocomplement in the sense of Stenström [7], i.e., an element $c \in L$ is *a pseudocomplement* of an element $a \in L$ if $a \wedge c = 0$ and c is maximal with this property. We shall mention and use in the sequel several results from [7], mainly contained in the third chapter, section 6.

We first prove two useful caracterizations

THEOREM 1. *The element $c \in L$ is a pseudocomplement of b in L if and only if $b \wedge c = 0$ and $b \vee c$ is essential in $[c, 1]$.*

Proof. Indeed, if $b \wedge c = 0$, for every $d \in [c, 1]$, $d \neq c$ the following statements are equivalent:

(i) $b \wedge d \neq 0$; (ii) $b \wedge d \not< c$; (iii) $c \lneq c \vee (b \wedge d) = (b \vee c) \wedge d$.

THEOREM 2. *Suppose L is pseudocomplemented. The element $c \in L$ is a pseudocomplement of b in L if and only if $b \wedge c = 0$, $b \vee c$ is essential in L and c is essentially closed.*

Proof. The conditions are clearly necessary (cf. [7]). Conversely, using theorem 1, we shall prove that they imply $b \vee c$ is essential in $[c, 1]$. Using the equivalent condition (iii) above, suppose d is an element in L such that $c \lneq d$ and $(b \vee c) \wedge d = c$. The element c being essentially closed, let $d' \in L$ be such that $0 \lneq d' \leqslant d$ and $c \vee d' = 0$. Then $0 = c \wedge d' = ((b \vee c) \wedge d) \wedge d' = (b \vee c) \wedge d'$. This contradicts the condition $b \vee c$ is essential in L.

We first apply these results to the lattice $L(A)$ of subgroups of an abelian group A, lattice which has all the properties we need. Traditionally, we say that a subgroup C is *B-high* if C is a pseudocomplement of B in $L(A)$.

Corollary. *If A is an abelian group and B, C are disjoint subgroups of A (i.e., $B \cap C = 0$) then the following conditions are equivalent: (1) C is B-high; (2) $B + C/C$ is essential in A/C. (3) $A/B + C$ is torsion and the socle $S(A/C) \leqslant B + C/C$.*

Remark. The implication: „C is B-high $\Rightarrow A/B + C$ is a torsion group" is known from a long time ([2] and [6]), the condition (3) above gives an adequate supply for an equivalence.

Remark. Using (3) one can easily improve the lemma 9.8 [1]: if C is B-high and n is a square-free integer then $a \in A$ and $na \in C$ implies $a \in B + C$.

2. Mutual pseudocomplements. If L is a modular lattice with 0 and 1 and $b \wedge c = 0$, $b \vee c = 1$ then b is a pseudocomplement of c and c is a pseudocomplement of b. We shall now introduce a new notion, a natural generalization of the situation above.

Definition. In a modular lattice L with 0 and 1 the elements $b, c \in L$ are mutual pseudocomplements if b is a pseudocomplement for c and c is a pseudocomplement of b.

The following for results are taken for the sake of completeness from [7] in a pseudo-complemented lattice:

(*a*) *If b is a pseudocomplement of a in L, then there exists a pseudocomplement c of b in L such that $a \leqslant c$.*

(*b*) *If b is a pseudocomplement of a in L, c is a pseudocomplement of b in L and $a \leqslant c$, then c is a maximal essential extension of a.*

Lemma. *Each pseudocomplement (or, equivalently, essential closed element) has at least one mutual pseudocomplement.*

Proof. One easily checks that in (*b*), b is also a pseudocomplement for c, because $a \wedge b' \neq 0$ for $b \lneq b'$ implies $b' \wedge c \neq 0$ since $a \leqslant c$.

Using theorem 2, we easily get

THEOREM 3. *Suppose L is a modular, pseudocomplemented lattice with 0 and 1. The elements $b, c \in L$ with $b \wedge c = 0$ are mutual pseudocomplements if and only if they are essentially closed and $b \vee c$ is essential in L.*

Corollary. *Two essential closed, disjoint subgroups B, C of an abelian group A are mutual high if and only if $A/B + C$ is a torsion group and $S(A) \leqslant B + C$.*

The mutual high subgroups actually generalizes the notion of direct sum in an abelian groups. For instance, in $\mathbf{Z}_8 \oplus \mathbf{Z}_2 = \langle a, b \rangle$ where $8a = 2b = 0$ the subgroups $B_1 = \{0, b\}$ and $B_2 = \{0, 2a + b, 4a, 6a + b\}$ are mutual high and B_2 is not a direct summand. It is also known that mutual pseudocomplements (being essentially closed subgroups) are always direct summands in a quasi-injective abelian group. Next, we show that in an arbitrary abelian group there are subgroups having all the mutual high subgroups direct summands.

THEOREM 4. *Let A^1 be the first Ulm subgroup of A. If B is mutual high with A^1 then $A = B \oplus A^1$.*

Proof. If B is A^1-high then B is pure in A([4] or [1], chap. 26, ex. 12). Further, if B is pure in A then $B + A^1/A^1$ is pure in A/A^1 ([1], 26, ex. 6). But A^1 being B-high, using theorem 1, $B + A^1/A^1$ is also essential in A/A^1. Hence $B + A^1/A^1 = A/A^1$ and $B + A^1 = A$. If A is a torsion group a little bit more can be said:

THEOREM 5. *Let A be a torsion abelian group and B a subgroup of A^1. If C is mutual high with B then $B \oplus C = A$.*

Proof. Similarly with the proof of theorem 4, we only need to check that C pure in A ([5]) implies $C + B/B$ pure in A/B. Suppose $c + B = na + B$ for an integer n. Hence $c - na \in B \leqslant A^1 = \bigcap_m mA \leqslant nA$ so that $c \in nA$ and $c \in C \cap nA$. Finally, C being pure in A we have $c \in nC$ and $c + B \in n(C + B/B)$.

Corollary. *If A is a torsion abelian group, any essential closed subgroup of A^1 is a direct summand of A.*

3. Miscellanea. The following two results are elementary:

PROPOSITION 6. *Suppose f is an endomorphism of an abelian group A. Then:*

(i) $\ker(f) \cap \operatorname{im}(f) = 0 \Leftrightarrow \ker(f) = \ker(f^2)$;

(ii) $\ker(f) + \operatorname{im}(f) = A \Leftrightarrow \operatorname{im}(f) = \operatorname{im}(f^2)$;

(iii) $\ker(f) \oplus \operatorname{im}(f) = A \Leftrightarrow f$ *is left and right regular in the ring End (A).*

In this context we finally prove

PROPOSITION 7. *If f is an endomorphism of an abelian group A, then $\ker(f)$ is $\operatorname{im}(f)$-high if and only if $\ker(f) = \ker(f^2)$ and $\operatorname{im}(f^2)$ is essential in $\operatorname{im}(f)$.*

Proof. If $a \in \operatorname{im}(f)$, $a \neq 0$ and $\ker(f)$ is $\operatorname{im}(f)$-high we must show that $\operatorname{im}(f^2) \cap \langle a \rangle \neq 0$. Now, if $a = f(a')$ we have $a' \notin \ker(f)$ and hence $(\langle a' \rangle + \ker(f)) \cap \operatorname{im}(f) \neq 0$. Thus, we have $n \in \mathbf{Z}$ and a'', $a_1 \in A$ such that $na' + a'' = f(a_1) \neq 0$ with $a'' \in \ker(f)$. So $f(na' + a'') = f^2(a_1) \neq 0$ because $\ker(f) = \ker(f^2)$. But $na = f^2(a_1) \neq 0$ and hence $\operatorname{im}(f^2) \cap \langle a \rangle \neq 0$. Conversely, if B is a subgroup of A, $\ker(f) \lneqq B$ and $\operatorname{im}(f^2)$ is essential in $\operatorname{im}(f)$, we must show that $B \cap \operatorname{im}(f) \neq 0$. If $a \in B$ is such that $f(a) \neq 0$ we must have by hypothesis $\operatorname{im}(f^2) \cap \langle f(a) \rangle \neq 0$. Hence we can get $n \in \mathbf{Z}$ and $a' \in A$ such that $f^2(a') = nf(a) = f(na) \neq 0$. So $f(a') - na \in \ker(f) < B$ and hence $f(a') \in B$. But $f(a') \neq 0$ because $\ker(f) = \ker(f^2)$ and $f^2(a') \neq 0$, and so $B \cap \operatorname{im}(f) \neq 0$.

Using again theorem 2, we finally have

Corollary. *Suppose f is an endomorphism of an abelian group A such that $\ker(f) = \ker(f^2)$. The subgroups $\ker(f)$ and $\operatorname{im}(f)$ are mutually high if and only if $\operatorname{im}(f^2)$ is essential in $\operatorname{im}(f)$ and $\operatorname{im}(f)$ is essentially closed.*

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