Distributivity and IM-lattices

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Abstract

Notice that in the diamond (i.e., one of the two 5-element nondistributive lattice) the intersection of the (three) maximal elements is not irredundant, and a lattice is not distributive if it contains a diamond. Hence, connections between the distributivity of a lattice and the irredundancy of the intersection of its family of maximal elements seem plausible.

In this paper, the authors prove that, under natural hypothesis, distributivity is equivalent with certain conditions on maximal elements. Applications to the distributivity of the lattice of all ideals of a semiprimitive ring with identity are given.

1 Introduction

The authors of [4] prove that a finite group G is cyclic if and only if for each subgroup H of G, $G =< H, H_0 >$ holds, H_0 denoting the intersection of all maximal subgroups of G which do no contain H.

G. L. Walls (unpublished) noticed that in the above characterization it suffices to take only the maximal subgroups H. Hence (via a well-known theorem of Ore), the lattice of all subgroups of a finite group is distributive if and only if for each maximal subgroup M of G, $G =< M, M_0 >$. Thus, for

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this particular type of lattices, distributivity is equivalent to this property. The authors of [4] wrote: "to what extent this is a more general phenomenon and which are the lattices in which it could still be detected is an open problem".

In what follows we consider several conditions in lattices (some of them considered for finite or soluble groups in [4], [8], [5] and [3]), and prove a characterization of distributivity using the family of all the maximal elements.

In order to increase the number of possible applications of these results we have deliberately avoided other natural conditions such as the modularity and the compactly generation of the lattice.

Applications for the lattice of all ideals of a semiprimitive ring with identity are given.

We have preferred the term 'maximal' element and 'intersection' instead of 'dual atom' respectively 'meet'. All the lattices we consider have 1 (the largest element). We use the quotient sublattice notation from [2], we denote by $\mathcal{M}$ the set of all the maximal elements in a lattice L , and for an element $a \in L$, by $\mathcal{D} = \{m \text{ maximal in } L | a \leq m\}$, $\mathcal{N} = \{m \text{ maximal in } L | a \not\leq m\}$ and $a_0 = \bigwedge \mathcal{N}$. As special case, for any maximal element m, $m_0 = \bigwedge(\mathcal{M} - \{m\})$ (because $m \not\leq m'$ and $m \neq m'$ are equivalent for every maximal elements m, m'). We denote by $r(L)$ the radical of a lattice, the intersection of all maximal elements in L. The lattices with each element contained in a maximal element are called *dually atomic* (and *relatively dually atomic* if each quotient sublattice is dually atomic). A subset X of elements in a lattice with 1 is called *meet-independent* if for every $a \in X$, the equality $a \vee (\bigwedge(X - \{a\})) = 1$ holds. The intersection $\bigwedge X$ is called *irredundant* if for every $a \in X$, $\bigwedge X \neq \bigwedge(X - \{a\})$.

2 The conditions A, B and C

In this section we deal with the following conditions (considered for finite groups in [4]):

condition **A***:* for each $a \in L$ and for every maximal element $m \in L$, $a \leq m$ and $a_0 \not\leq m$ are equivalent.

condition **B***:* $m \vee m_0 = 1$ holds for every maximal element $m \in L$.

condition **C**: $a \vee a_0 = 1$ holds for every $a \in L$.

Notice that in a complete lattice L the join of an empty family of elements is 1 (this follows a well-known set-theoretic convention). For instance, if $a \leq m$ holds for every maximal element in L, then $a_0 = \bigwedge\{m_1 \text{ maximal } | a \nleq m_1\} = \bigwedge \emptyset = 1$ (and trivially, $a \vee a_0 = 1$). With the above notation the equalities $\mathcal{D} \cap \mathcal{N} = \emptyset$, $\mathcal{D} \cup \mathcal{N} = \mathcal{M}$ hold. Also notice that $a_0 \leq m$ for every $m \in \mathcal{N}$.

Remark. If **C** holds then the sublattice $1/r(L)$ is complemented (indeed, for every $a \in L$ we have $a \leq \bigwedge \mathcal{D}$ and $a_0 = \bigwedge \mathcal{N}$. Hence $a \wedge a_0 \leq \bigwedge \mathcal{M} = r(L)$ and the reversed inequality also holds if $a \in 1/r(L)$).

Proposition 2.1 *In an arbitrary lattice, condition* **C** *implies any of the equivalent conditions* **A** *and* **B**. *In a dually atomic lattice, conditions* **A**, **B** *and* **C** *are equivalent.*

Proof. In an arbitrary lattice L, obviously **C** implies **B.**

Further, using the definition of a_0, it should be observed that $a_0 \nleq m \Rightarrow a \leq m$ is true in any lattice. Therefore **A** actually is equivalent to $a \leq m \Rightarrow a_0 \nleq m$. Denoting by $\mathcal{D}_0 = \{m \text{ maximal in } L | a_0 \leq m\}$ and by $\mathcal{N}_0 = \{m \text{ maximal in } L | a_0 \nleq m\}$, it follows immediately that **A** is equivalent with $\mathcal{D} = \mathcal{N}_0$, and hence also with $\mathcal{D}_0 = \mathcal{N}$ (indeed $\mathcal{N}_0 \subseteq \mathcal{D}$ holds for every $a \in L$).

A $\Rightarrow$ **B** Take $a = m$. Then $m_0 \nleq m$ and so $m < m \vee m_0$ implies $m \vee m_0 = 1$.

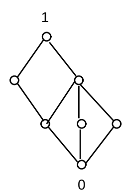
B $\Rightarrow$ **A** Suppose **A** does not hold. Then there is an element $a \in L$ and a maximal element m such that $a \leq m$ and $a_0 \leq m$ (i.e., $\mathcal{N}_0 \subset \mathcal{D}$). Hence $m \notin \mathcal{N}$ and so $\mathcal{N} \subseteq \mathcal{M} - \{m\}$. Consequently, $m_0 = \bigwedge \mathcal{M} - \{m\} \leq \bigwedge \mathcal{N} = a_0$. Finally, $m \vee m_0 \leq m \vee a_0 = m$ and **B** does neither hold.

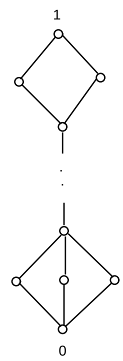
A $\Rightarrow$ **C** Assume $a \vee a_0 \neq 1$ for an element $a \in L$. The lattice L being dually atomic consider a maximal element $m \in L$ such that $a \vee a_0 \leq m$. Then $a \leq m$ and $a_0 \leq m$ so that **A** does not hold. $\square$

Example. Each compact lattice, by a classical result of Krull, is dually atomic, so that in a compact lattice conditions **A**, **B** and **C** are equivalent. In particular, these conditions are equivalent in every finite lattice.

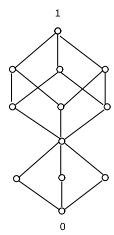
Notice that condition **B** holds in a distributive lattice L with finitely many maximal elements but the converse is not true. In the sequel we give examples of finite nondistributive lattices which satisfy condition **B**.

Obviously $m \vee m_0 = 1$ (trivially) holds in every lattice with exactly 2 maximal elements. Hence, if we take the 4-element lattice with 2 incomparable elements and adjoin below the diamond (i.e., the 5-element nondistributive lattice), we obtain counterexamples of arbitrary dimension (see the diagrams below).





If we care for examples with n maximal elements we consider the distributive lattice of all subsets of an $(n+1)$-element set and adjoin again the diamond below. The following diagram, an 8-element distributive lattice with 3 maximal elements, describes the case $n = 3$.



Finally we prove a converse for the implication mentioned above, "condition **B** holds in every distributive lattice with finitely many maximal elements".

Proposition 2.2 *The sublattice generated by a finite meet-independent family of n maximal elements is isomorphic to the Boole algebra of all subsets of an n-element set.*

Proof. Notice that a *family of maximal elements is meet-independent if and only if it has an irredundant intersection*. Indeed, $\bigwedge X = a \wedge (\bigwedge(X - \{a\})) \neq \bigwedge(X - \{a\})$ is equivalent to $\bigwedge(X - \{a\}) \not\leq a$, or, $a < a \vee (\bigwedge(X - \{a\}))$ and hence $a \vee (\bigwedge(X - \{a\})) = 1$, a being a maximal element.

By definition, condition **B** holds exactly if the set of all maximal elements $\mathcal{M}$ is meet-independent, or, according to the previous remark, when the intersection $\bigwedge \mathcal{M}$ is irredundant.

The diagram which represents the sublattice generated by all maximal elements has (excepting 1) n different levels, corresponding to the two maximal element intersections, the three maximal element intersections, etc.

First observe that there are no comparable intersections of either two maximal elements.

Indeed, if, for example, $m_1 \wedge m_2 \leq m_3 \wedge m_4$ then $(\bigwedge\limits_{i \neq 3} m_i) \vee m_3 = m_3$ because $(\bigwedge\limits_{i \neq 3} m_i) \leq m_1 \wedge m_2 \leq m_3$, contradicting **B**.

Similarly, the three maximal element intersections are non-comparable, ..., the $n-1$ maximal element intersections are n non-comparable elements and the intersection of all maximal elements (i.e., $r(L)$) is the smallest element of the sublattice.

Hence, the sublattice is isomorphic to the lattice of all subsets of an n-element set, and to the corresponding Boole algebra. $\square$

Corollary 2.1 *The sublattice generated by a finite meet-independent family of maximal elements is distributive.* $\square$

3 IM-lattices

A lattice L is called an *IM-lattice* if every element $a \in 1/r(L)$ is an intersection of maximal elements (or equivalently, every element a is the intersection of all maximal elements which contain a). Equivalently, $r(1/a) = a$ holds for each $a \geq r(L)$.

The pentagon (i.e., the 5-element nonmodular lattice) is an example of a dually atomic lattice with $r(L) = 0$ and finitely many maximal elements which is not an IM-lattice.

A lattice L with 1 is called *weakly join-complemented* ([5]) if for every pair of elements x, $y \in L$ such that $x < y$ there is an element $z \in L$ so that $x \vee z \neq 1$ and $y \vee z = 1$.

A lattice L with 1 is called *LO-lattice* ([3]) if for every pair of elements x, $y \in L$ such that $x \prec y$ (i.e., x is covered by y) there is a maximal element $m \in L$ such that $x = y \wedge m$.

Remarks. - A complement z of y in $1/x$ clearly satisfies $x \vee z \neq 1$ and $y \vee z = 1$, and so *each relatively complemented lattice is weakly join-complemented.*

- *Each IM-lattice is weakly join-complemented.* Indeed, let $x < y$ be elements in an IM-lattice L. If $x = \bigwedge \mathcal{D}_x$ and $y = \bigwedge \mathcal{D}_y$ with $\mathcal{D}_x = \{m$ maximal in $L | x \leq m\}$ and $\mathcal{D}_y$ has similar meaning, then $\mathcal{D}_y \subset \mathcal{D}_x$ and for $m \in \mathcal{D}_x - \mathcal{D}_y$, we obtain $x \leq m$ and $y \not\leq m$. Hence $x \vee m = m \neq 1$, $y \vee m = 1$ and the lattice is weakly join-complemented.

The converse of this last implication seems to need severe restrictions on the lattice (e.g., cycle generated).

An example of a finite weakly join-complemented lattice which is not an IM-lattice is the direct product of two diamonds (each being a simple lattice).

Proposition 3.1 (Deaconescu) *Each IM-lattice is an LO-lattice. In a relatively dually atomic lattice the converse holds.*

Proof ([3]). Suppose that in an IM-lattice L there are elements $x \prec y$ such that $x \neq y \wedge m$ for every maximal element m in L. Then, for each maximal element m in L such that $x \leq m$, $y \leq m$ also holds (otherwise $x < y \wedge m < y$ and y does not cover x), i.e., $\mathcal{D}_x \subseteq \mathcal{D}_y$. Hence $y = \bigwedge \mathcal{D}_y \leq \bigwedge \mathcal{D}_x = x$, which contradicts $x < y$.

Conversely, in an LO-lattice suppose that $a < \bigwedge \mathcal{D}_a$ holds for an element $a \in L$. If the lattice is relatively dually atomic, there is a maximal element m' in $(\bigwedge \mathcal{D}_a)/a$, and so a maximal element m in L, such that $m' = (\bigwedge \mathcal{D}_a) \wedge m$. Hence $\bigwedge \mathcal{D}_a \not\leq m$ (otherwise $\bigwedge \mathcal{D}_a = m'$) and further, $a \not\leq m$ (otherwise $m \in \mathcal{D}_a$ and $\bigwedge \mathcal{D}_a \leq m$) which contradicts $a \leq m' \leq m$. $\square$

Corollary 3.1 *A finite lattice is an IM-lattice if and only if it is an LO-lattice.* $\square$

Using the previous Remark and Proposition we obtain

Corollary 3.2 *Every relatively dually atomic LO-lattice is weakly join-complemented.* $\square$

Resuming the final results of the previous section, notice that in an IM-lattice L, the sublattice generated by a finite meet-independent family of maximal elements is $1/r(L)$. Obviously this is L if $r(L) = 0$ also holds. Then

Corollary 3.3 *An IM-lattice with $r(L) = 0$ which satisfies condition* **B** *is distributive.* $\square$

Our first main result is

Theorem 3.1 *In an IM-lattice L with $r(L) = 0$ and a finite family $\mathcal{M}$ of maximal elements the following conditions are equivalent:*
L is distributive,
$\mathcal{M}$ is a meet-independent family,
$\bigwedge \mathcal{M}$ is irredundant,
condition **A***, or*
condition **B***.* $\square$

Corollary 3.4 *A finite IM-lattice with $r(L) = 0$ is distributive if and only if it satisfies any of the conditions* **A***,* **B** *or* **C***.* $\square$

We leave to the reader the connections with weakly join-complemented lattices and LO-lattices.

4 The condition D

In this section we consider *condition* **D**: $a = a_{00}$ holds for every $a \in 1/r(L)$. Here a_{00} denotes $(a_0)_0$.

Proposition 4.1 *A lattice L satisfies condition* **D** *if and only if it is an IM-lattice which satisfies condition* **B***.*

Proof. A lattice which satisfies condition **D** is an IM-lattice ($a = a_{00}$ is an intersection of maximal elements). Further, condition **D** implies condition **B**. Indeed, if $a = a_{00}$ holds for every element of $1/r(L)$, this is also true for the maximal elements of L. Let m be an arbitrary maximal element of L. Then $m_0 = \bigwedge(\mathcal{M} - \{m\})$ and $m_0 \leq m'$ holds for every $m' \in \mathcal{M} - \{m\}$. Two cases arise: $m_0 \not\leq m$ and thus $m = m_{00}$, and, $m_0 \leq m$ so that $m_{00} = 1 \neq m$ (as empty intersection of maximal elements). Finally the condition $m_0 \not\leq m$ is equivalent to $r(L) = \bigwedge \mathcal{M}$ being an irredundant intersection (that is, for every maximal element m, $r(L) \neq \bigwedge(\mathcal{M} - \{m\})$), which is equivalent to condition **B**.

Conversely, the conditions **A** and **B** being equivalent (Proposition 2.1), we show that if condition **A** holds in an IM-lattice then also condition **D** holds.

First mention a simple assertion: for every element $a \in L$ the following holds

$$a \leq \bigwedge \mathcal{D} \leq \bigwedge \mathcal{N}_0 = a_{00} \quad (1) \ .$$

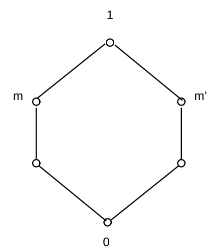
Indeed, with our previous notation, $a_{00} = \bigwedge \mathcal{N}_0$ and so $a_{00} \leq m$ for every $m \in \mathcal{N}_0$. Owing to the equalities $\mathcal{D}_0 \cap \mathcal{N}_0 = \emptyset$, $\mathcal{D}_0 \cup \mathcal{N}_0 = \mathcal{M}$, by the definition of $\mathcal{D}_0$, we have $\mathcal{N} \subseteq \mathcal{D}_0$ and consequently $\mathcal{N}_0 \subseteq \mathcal{D}$.

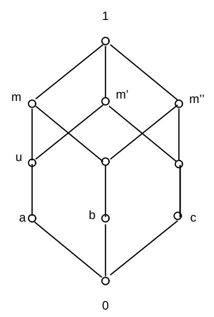
Finally, as previously noticed, condition **A** is equivalent to $\mathcal{D} = \mathcal{N}_0$ and this is equivalent with $\mathcal{D}_0 = \mathcal{N}$. Hence above, in (1), we have only equalities and condition **D** holds. $\square$

Corollary 4.1 (of the proof) *If* $a = a_{00}$ *then* $a = \bigwedge \mathcal{D}$. $\square$

Remark. More precisely, *if condition* **B** *holds, every intersection of maximal elements satisfies* condition **D**. Indeed, this follows at once using the fact that, for every subset $\mathcal{M}'$ of maximal elements, $\bigwedge \mathcal{M}'$ is irredundant (or equivalently, $\mathcal{M}'$ is meet-independent), together with $\bigwedge \mathcal{M}$ (and so, if $a = \bigwedge \mathcal{M}'$, then $\mathcal{M}' = \mathcal{D}$). This yields another proof for the second part of the previous Proposition.

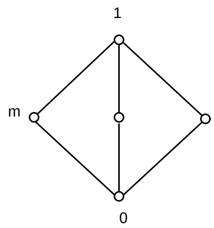
Remark. The result given in the previous Proposition cannot be improved. Indeed, condition **B** does not imply IM (for example take the 3-element chain) nor does condition **B** together with $r(L) = 0$. The following two diagrams are suitable examples.





Conversely, an IM-lattice does not necessarily satisfy condition **B** (the diamond, i.e. the 5-element non-distributive lattice - the lattice of all subgroups of the Klein group - or, a 6-element lattice, isomorphic with the lattice of all subgroups of the symmetric group S_3).

Moreover, condition **B** does not imply condition **D** (take the element a in the previous diagram; $a_0 = m''$ and $a_{00} = m \wedge m' = u > a$), nor does IM imply condition **D** (again the diamond: $m_{00} = 1 \neq m$ in the next diagram).

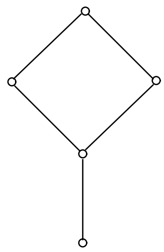


Finally, our second main result is

Theorem 4.1 *A lattice with finitely many maximal elements and* $r(L) = 0$ *satisfies the condition* **D** *if and only if it is a distributive IM-lattice.*

Proof. One has only to use Corollary 3.1 and Proposition 4.1. □

Remark. Again, this result cannot be improved. The next diagram represents a distributive lattice which is not IM.



5 Examples and applications.

Call *DA-groups* the abelian groups which have a dually atomic lattice of subgroups (i.e., each proper subgroup is contained in a maximal subgroup). This is a rather restrictive condition. Indeed, one can prove that *a group G is a DA-group if and only if all its p-components are bounded and $G/T(G)$ is of finite torsion-free rank and of reduced Richman type.*

Next, using the characterization of the soluble IM-groups given in [8], notice that *the abelian IM-groups are exactly the elementary groups.*

Indeed, for an abelian (IM-groups are torsion) group, if there is an elementary Hall subgroup with elementary quotient, this subgroup has to be a direct summand (the corresponding extension is splitting) and so, the group is elementary; conversely, the subgroup lattice of an abelian IM-group is modular and complemented and hence also relatively complemented. We can apply the characterization given in [5] and the first Remark from Section 3.

Finally, call *A-groups* the groups which have maximal groups (i.e., we discard the divisible groups) and satisfy condition **A**. We can prove that *there are no mixed A-groups, the torsion A-groups are the prime power order cyclic groups, and, the torsion-free A-groups are the rank 1 groups whose types have no infinity entry and have infinitely many non-zero entries.*

The lattice $L(R)$ of all ideals of an associative ring R with identity being dually atomic, applications of our results are to be expected. Recall that R is called *semiprimitive* if its Jacobson radical $J(R)$ vanishes.

Using Theorem 3.1 and Theorem 4.1 we obtain at once

Corollary 5.1 *The lattice $L(R)$ of all ideals of a semiprimitive ring with identity, with finitely many maximal ideals, such that every ideal is an intersection of maximal ideals is distributive if and only if it is satisfies any of the conditions* **A** *or* **B**. □

Corollary 5.2 *The lattice $L(R)$ of all ideals of a semiprimitive ring with identity and with finitely many maximal ideals satisfies condition* **D** *if and only if it is distributive and every ideal is an intersection of maximal ideals.* □

Corollary 5.3 *The lattice $L(R)$ of all ideals of a finite semiprimitive ring with identity such that every ideal is an intersection of maximal ideals is distributive if and only if it satisfies any of the conditions* **A**, **B** *or* **C**. □

Corollary 5.4 *The lattice $L(R)$ of all ideals of a finite semiprimitive ring with identity is distributive if and only if it satisfies condition* **D**. □

Following Fuchs [6], a commutative ring with identity is called *arithmetical* if the ideals form a distributive lattice. Several characterizations of arithmetical rings using maximal ideals (but also the corresponding local (generalized) quotient rings) are given in [6] and [7]. The semiprimitive case is also partially discussed. As for now, the authors do not have any link between these apparently similar characterizations and the applications obtained above.

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