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A nil-clean 2×2 matrix over the integers which is not clean

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While any nil-clean ring is clean, the last eight years, it was not known whether nil-clean elements in a ring are clean. We give an example of nil-clean element in the matrix ring $\mathcal{M}_2(\mathbf{Z})$ which is not clean.

Keywords: Nil-clean; Diophantine equation; Pell equation; unit regular.

1. Introduction

Let R be a ring with identity. An element $a \in R$ is called *unit-regular* if $a = bub$ with $b \in R$ and a unit u in R, *clean* if $a = e + u$ with an idempotent e and a unit u, and *nil-clean* if $a = e + n$ with an idempotent e and a nilpotent n. A ring is *unit-regular* (or *clean*, or *nil-clean*) if all its elements are so. In [2], it was proved that *every unit-regular ring is clean*. However, in [5], it was noticed that this implication, *for elements*, fails. In the paper, plenty of unit-regular elements which are not clean are found among 2×2 matrices of the type $\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right]$ with integer entries.

While it is easy to prove that *any nil-clean ring is also a clean ring*, the question Whether *nil-clean elements are clean*? was left open (see [3] and restated in [4]) for some seven years. In this note, we answer in the negative this question.

2. Preliminaries

As this was done (in a special case) in [5], we investigate elements in the 2×2 matrix ring $\mathcal{M}_2(\mathbf{Z})$. Since $\mathbf{Z}$ and direct sums of $\mathbf{Z}$ are not clean (not even exchange rings), it makes sense to look for elements which are not clean in this matrix ring.

We first recall some elementary facts.

Let R be an integral domain and $A \in \mathcal{M}_n(R)$. Then A is a zero divisor if and only if $\det A = 0$. Therefore idempotents (excepting the identity matrix) and nilpotents have zero determinant.

For $A \in \mathcal{M}_n(R), rk(A) < n$ if and only if $\det A$ *is a zero divisor in R. A matrix A is a unit in $\mathcal{M}_n(R)$ if and only if* $\det A \in U(R)$. Thus, *the units* in $\mathcal{M}_2(\mathbf{Z})$ are the 2×2 matrices of $\det = \pm 1$.

Lemma 1. *Nontrivial idempotents in $\mathcal{M}_2(\mathbf{Z})$ are matrices $\left[\begin{smallmatrix} \alpha+1 & u \\ v & -\alpha \end{smallmatrix}\right]$ with $\alpha^2 + \alpha + uv = 0$.*

Proof. One way follows by calculation. Conversely, notice that excepting I_2, such matrices are singular. Any nontrivial idempotent matrix in $\mathcal{M}_2(\mathbf{Z})$ has rank 1. By Cayley–Hamilton Theorem, $E^2 - \mathrm{tr}(E)E + \det(E)I_2 = 0$. Since $\det(E) = 0$ and $E^2 = E$ we obtain $(1 - \mathrm{tr}(E)) \cdot E = 0_2$ and so, since there are no zero divisors in $\mathbf{Z}$, $\mathrm{tr}(E) = 1$. □

Lemma 2. *Nilpotents in $\mathcal{M}_2(\mathbf{Z})$ are matrices $\left[\begin{smallmatrix} \beta & x \\ y & -\beta \end{smallmatrix}\right]$ with $\beta^2 + xy = 0$.*

Proof. One way follows by calculation. Conversely, just notice that nilpotent matrices in $\mathcal{M}_2(\mathbf{Z})$ have the characteristic polynomial t^2 and so have trace and determinant equal to zero. □

Therefore, *the set of all the nil-clean matrices* in $\mathcal{M}_2(\mathbf{Z})$, which use a nontrivial idempotent in their nil-clean decomposition, is

$$\left\{ \begin{bmatrix} \alpha+\beta+1 & u+x \\ v+y & -\alpha-\beta \end{bmatrix} \,\middle|\, \alpha, \beta, u, v, x, y \in \mathbf{Z}, \alpha^2 + \alpha + uv = 0 = \beta^2 + xy \right\}.$$

Remark. (1) Nil-clean matrices in $\mathcal{M}_2(\mathbf{Z})$ which use a nontrivial idempotent, have the trace equal to 1. Otherwise, this is 2 or 0.

(2) Since only the absence of nonzero zero divisors is (essentially) used, the above characterizations hold in any integral domain.

It is easy to discard the triangular case.

Proposition 3. *Upper triangular nil-clean matrices, which are neither unipotent nor nilpotent, are idempotent, and so (strongly) clean.*

Proof. Such upper triangular idempotents are $\left[\begin{smallmatrix} \alpha+1 & u \\ 0 & -\alpha \end{smallmatrix}\right]$ with $-\det = \alpha^2 + \alpha = 0$, so have $\alpha \in \{-1, 0\}$, that is, $\left[\begin{smallmatrix} 1 & u \\ 0 & 0 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} 0 & u \\ 0 & 1 \end{smallmatrix}\right]$. Upper triangular nilpotents have the form $\left[\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right]$, and so upper triangular nil-clean matrices have the form $\left[\begin{smallmatrix} 1 & u \\ 0 & 0 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} 0 & u \\ 0 & 1 \end{smallmatrix}\right]$. As noticed before, these are idempotent. □

Equations with more than one independent variable and with integer coefficients for which integer solutions are desired are called *Diophantine equations*. The ones we use in the sequel have the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

where a, b, c, d, e, f and are integers, i.e. *general inhomogeneous equations of the second degree with two unknowns.*

Denote (see [1] or [6]) $D =: b^2 - 4ac$, $g =: \gcd(b^2 - 4ac; 2ae - bd)$ and $\Delta =: 4acf + bde - ae^2 - cd^2 - fb^2$. Then the equation reduces to

$$-\frac{D}{g}Y^2 + gX^2 + 4a\frac{\Delta}{g} = 0$$

which (if $D > 0$) is a general Pell equation. Here $Y = 2ax + by + d$ and $X = \frac{D}{g}y + \frac{2ae-bd}{g}$. Notice that this equation may be also written $-DY^2 + X^2 + 4a\Delta = 0$ replacing X by gX (and so $X = Dy + 2ae - bd$).

3. The General Case

In order to find a nil-clean matrix in $\mathcal{M}_2(\mathbf{Z})$ which is not clean, we need integers $\alpha, \beta, u, v, x, y$ with $\alpha^2 + \alpha + uv = 0 = \beta^2 + xy$, such that for every $\gamma, s, t \in \mathbf{Z}$, with $\gamma^2 + \gamma + st = 0$, the determinant

$$\det\left[\begin{bmatrix} \alpha+\beta-\gamma & u+x-s \\ v+y-t & -\alpha-\beta+\gamma \end{bmatrix}\right]$$
$$= -(\alpha+\beta-\gamma)^2 - (u+x-s)(v+y-t) \notin \{\pm 1\}.$$

That is, subtracting any idempotent $\left[\begin{smallmatrix} \gamma+1 & s \\ t & -\gamma \end{smallmatrix}\right]$ from $\left[\begin{smallmatrix} \alpha+1 & u \\ v & -\alpha \end{smallmatrix}\right] + \left[\begin{smallmatrix} \beta & x \\ y & -\beta \end{smallmatrix}\right]$, the result should not be a unit in $\mathcal{M}_2(\mathbf{Z})$.

Remark. Notice that above we have excepted the trivial idempotents. However, this will not harm since, in finding a counterexample, we ask for the nil-clean example not to be idempotent, nilpotent nor unit (and so not unipotent).

In the sequel, to simplify the writing, the following notations will be used: first, $m := 2\alpha + 2\beta + 1$ (m is odd and so nonzero) and $n := (u+x)(v+y) + (\alpha+\beta)^2 + 1$ and second, $r := \alpha + \beta$ and $\delta := r^2 + r + (v+y)(u+x)$. Then $m = 2r + 1$, $n = (u+x)(v+y) + r^2 + 1 = \delta - r + 1$. This way an arbitrary nil-clean matrix which uses no trivial idempotents is now written $C = \left[\begin{smallmatrix} r+1 & u+x \\ v+y & -r \end{smallmatrix}\right]$ and $\delta = -\det C$. To simplify the wording such nil-clean matrices will be called *nontrivial nil-clean.*

Theorem 4. *Let* $C = \left[\begin{smallmatrix} r+1 & u+x \\ v+y & -r \end{smallmatrix}\right]$ *be a nontrivial nil-clean matrix and let* $E = \left[\begin{smallmatrix} \gamma+1 & s \\ t & -\gamma \end{smallmatrix}\right]$ *be a nontrivial idempotent matrix. With above notations,* $C - E$ *is invertible in* $\mathcal{M}_2(\mathbf{Z})$ *with* $\det(C - E) = 1$ *if and only if*

$$X^2 - (1 + 4\delta)Y^2 = 4(v+y)^2(2r+1)^2(\delta^2 + 2\delta + 2)$$

with $X = (2r+1)[-(1+4\delta)t + (2\delta+3)(v+y)]$ *and* $Y = 2(v+y)^2 s + (2r^2+2r+1+2\delta)t - (2\delta+3)(v+y)$. *Further,* $C - E$ *is invertible in* $\mathcal{M}_2(\mathbf{Z})$ *with* $\det(C-E) = -1$ *if and only if*

$$X^2 - (1+4\delta)Y^2 = 4(v+y)^2(2r+1)^2\delta(\delta-2)$$

with $X = (2r+1)[-(1+4\delta)t + (2\delta-1)(v+y)]$ *and* $Y = 2(v+y)^2 s + (2r^2+2r+1+2\delta)t - (2\delta-1)(v+y)$.

Proof. For given $\alpha, \beta, u, v, x, y$, $\det(C-E) = \pm 1$ amounts to a general inhomogeneous equation of the second degree with two unknowns, which we reduce to a canonical form, as mentioned in the previous section. Here are the details.

$$\begin{aligned} &-\gamma^2 - st - (\alpha+\beta)^2 + 2(\alpha+\beta)\gamma + (v+y)s + (u+x)t - (u+x)(v+y) \\ &\quad = (2\alpha+2\beta+1)\gamma + (v+y)s + (u+x)t - (u+x)(v+y) - (\alpha+\beta)^2 = \pm 1. \end{aligned}$$

The case $\det = 1$. Since $-m\gamma = (v+y)s + (u+x)t - (u+x)(v+y) - (\alpha+\beta)^2 - 1 = (v+y)s + (u+x)t - n$, we obtain from $(-m\gamma)^2 - m(-m\gamma) + m^2 st = 0$, the equation

$$\begin{aligned} &[(v+y)\mathbf{s} + (u+x)\mathbf{t} - n]^2 - m[(v+y)\mathbf{s} + (u+x)\mathbf{t} - n] + m^2 st = 0, \quad \text{or} \\ &(v+y)^2\mathbf{s}^2 + [2(v+y)(u+x) + m^2]\mathbf{st} + (u+x)^2\mathbf{t}^2 \\ &\quad - (m+2n)(v+y)\mathbf{s} - (m+2n)(u+x)\mathbf{t} + (m+n)n = 0. \end{aligned}$$

Thus, with the notations of the previous section

$$\begin{aligned} &\mathbf{a} = (v+y)^2, \quad \mathbf{b} = [2(v+y)(u+x) + m^2], \quad \mathbf{c} = (u+x)^2 \quad \text{and} \\ &\mathbf{d} = -(m+2n)(v+y), \quad \mathbf{e} = -(m+2n)(u+x), \quad \mathbf{f} = (m+n)n. \end{aligned}$$

Further

$$\begin{aligned} \mathbf{D} &= [2(v+y)(u+x) + m^2]^2 - 4(v+y)^2(u+x)^2 = m^4 + 4m^2(v+y)(u+x) \\ &= m^2[m^2 + 4(v+y)(u+x)], \end{aligned}$$

$\mathbf{2ae} - \mathbf{bd} = m^2(m+2n)(v+y)$ for $g = \gcd(D, 2ae - bd)$ (notice that $m^2 | g$) and

$$\begin{aligned} \mathbf{\Delta} &= 4acf + bde - ae^2 - cd^2 - fb^2 \\ &= 4(v+y)^2(u+x)^2(m+n)n + [2(v+y)(u+x) + m^2](m+2n)^2(v+y)(u+x) \\ &\quad - (v+y)^2(m+2n)^2(u+x)^2 - (u+x)^2(m+2n)^2(v+y)^2 \\ &\quad - (m+n)n[2(v+y)(u+x) + m^2]^2 \\ &= m^4[(v+y)(u+x) - (m+n)n]. \end{aligned}$$

The case $\det = -1$. Formally exactly the same calculation, but n is slightly modified: here $n' = (u+x)(v+y) + (\alpha+\beta)^2 - 1$, i.e. $n' := n - 2$.

These equations reduce to the canonical form

$$gX^2 - \frac{D}{g}Y^2 = -4a\frac{\Delta}{g}$$

with $D = m^2[m^2 + 4(v+y)(u+x)], g = \gcd(D, m^2(m+2n)(v+y)), a = (v+y)^2$ and $\Delta = m^4[(v+y)(u+x) - (m+n)n]$.

Since clearly $g = m^2g'$, in the above equation we can replace D and Δ by $\frac{D}{m^2}$ and $\frac{\Delta}{m^2}$ (and $g = \gcd(m^2 + 4(v+y)(u+x); (m+2n)(v+y)))$, that is $D = m^2 + 4(v+y)(u+x)$ and $\Delta = m^2[(v+y)(u+x) - (m+n)n]$.

Further, this amounts to $g^2X^2 - DY^2 = -4a\Delta$ and so we can eliminate g (by taking a new unknown: $X' = gX$). Hence we reduce to the equation

$$X^2 - [m^2 + 4(v+y)(u+x)]Y^2 = -4(v+y)^2m^2[(v+y)(u+x) - (m+n)n],$$

which we can rewrite as

$$X^2 - (1+4\delta)Y^2 = 4(v+y)^2(2r+1)^2(\delta^2 + 2\delta + 2).$$

Further, for det $= -1$, we obtain a similar equation replacing n by $n-2$, i.e. $n = \delta - r - 1$:

$$X^2 - (1+4\delta)Y^2 = 4(v+y)^2(2r+1)^2\delta(\delta - 2).$$

The linear systems in s and t corresponding to det $= 1$ and det $= -1$, are respectively:

$$\begin{cases} 2(v+y)^2s + (2r^2 + 2r + 1 + 2\delta)t - (2\delta + 3)(v+y) = Y, \\ (2r+1)[-(1+4\delta)t + (2\delta+3)(v+y)] = X \end{cases}$$

for det $= 1$ (here $-(2r+1)\gamma = (v+y)s + (u+x)t - n = (v+y)s + (u+x)t - \delta + r - 1$), and

$$\begin{cases} 2(v+y)^2s + (2r^2 + 2r + 1 + 2\delta)t - (2\delta - 1)(v+y) = Y, \\ (2r+1)[-(1+4\delta)t + (2\delta-1)(v+y)] = X \end{cases}$$

for det $= -1$ (here $-(2r+1)\gamma = (v+y)s + (u+x)t - n' = (v+y)s + (u+x)t - \delta + r + 1$). □

4. The Example

Since $1 + 4\delta \geq 1$ if $\delta \geq 0$, in this case, from the general theory of Pell equations, it is known that the equations emphasized in Theorem 4 have infinitely many solutions, and so we cannot decide whether all the linear systems corresponding to these equations, have (or not) integer solutions. However, if $\delta \leq -1$ then $1 + 4\delta < 0$ and we have elliptic type of Pell equations, which clearly have only finitely many integer solutions.

Take $r = 2$, $\delta = -57$ and $v + y = -7$, $u + x = 9$, that is, the matrix we consider is $\left[\begin{smallmatrix} 3 & 9 \\ -7 & -2 \end{smallmatrix}\right]$; $1 + 4\delta = -227$.

More precisely $\alpha = -1$, $\beta = 3$, $u = 0$, $v = -6$, $x = 9$ and $y = -1$, i.e. the nil-clean decomposition

$$\begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -6 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}.$$

The (elliptic) Pell equation which corresponds to a unit with det = 1 is $X^2 + 227Y^2 = 15.371.300$ with $X = 3(227t + 777)$ (we shall not need Y).

Since $X = 227(3t + 10) + 61$ we deduce $X^2 = 227k + 89$ for a suitable integer k. However, since $15.371.300 = 67.715 \times 227 - 5$ from the Pell equation we obtain $X^2 = 227l - 5$ (for a suitable integer l) and so there are no integer solutions.

As for the equation which corresponds to det $= -1$, $X^2 + 227Y^2 = 16.478.700$ with $X = 3(227t + 805)$. Analogously, $X = 227(3t + 10) + 145$ and $X^2 = 227p + 141$ (for some integer p). Since from the Pell equation $(16.478.700 = 72.593 \times 227 + 89)$ we obtain $X^2 = 227q + 89$ (for an integer q), again we have no integer solutions.

5. How the Example was Found

A deceptive good news is that both equations (in Theorem 4) are solvable (over $\mathbf{Z}$): the first equation admits the solutions $X = \pm(v + y)(2r + 1)(2\delta + 3)$ and $Y = \pm(v + y)(2r + 1)$, and the second equation admits the solutions: $X = \pm(v + y)(2r + 1)(2\delta - 1)$ and $Y = \pm(v + y)(2r + 1)$.

Therefore, the main problem which remains with respect to the solvability of the initial equations in s and t (γ is determined by s and t), is whether the linear systems above (in s and t), also have solutions (over $\mathbf{Z}$). Here is an analysis of this problem, just for the solutions given above.

For a unit with det = 1 we have four solutions: for $+X = +(v + y)(2r + 1)(2\delta + 3)$ we obtain $t = 0$. Then for $+Y = +(v + y)(2r + 1)$ we obtain $s = u + x + \frac{r^2+2r+2}{v+y}$ and $\gamma = -1$ and for $-Y = -(v + y)(2r + 1)$ we obtain $s = u + x + \frac{r^2+1}{v+y}$ and $\gamma = 0$. The corresponding clean decompositions are

$$\begin{aligned}\begin{bmatrix} r+1 & u+x \\ v+y & -r \end{bmatrix} &= \begin{bmatrix} 0 & u+x+\frac{r^2+2r+2}{v+y} \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} r+1 & -\frac{r^2+2r+2}{v+y} \\ v+y & -r-1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & u+x+\frac{r^2+1}{v+y} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} r & -\frac{r^2+1}{v+y} \\ v+y & -r \end{bmatrix}.\end{aligned}$$

Notice that $r^2 + 1$ and $r^2 + 2r + 2 = (r + 1)^2 + 1$ are nonzero.

For $-X = -(v + y)(2r + 1)(2\delta + 3)$ we obtain $t = (v + y)(1 + \frac{5}{1+4\delta})$ which is an integer if and only if $1 + 4\delta$ divides $5(v + y)$. However, this has to be continued with conditions on s.

For a unit with det $= -1$ we also have four solutions: for $+X = (v + y)(2r + 1)(2\delta - 1)$ we obtain $t = 0$. Then for $+Y = (v + y)(2r + 1)$ we obtain $s = u + x + \frac{r^2+2r}{v+y}$ and $\gamma = -1$ and for $-Y = -(v + y)(2r + 1)$ we obtain $s = u + x + \frac{r^2-1}{v+y}$ and $\gamma = 0$.

The corresponding clean decompositions are

$$\begin{bmatrix} r+1 & u+x \\ v+y & -r \end{bmatrix} = \begin{bmatrix} 0 & u+x+\frac{r^2+2r}{v+y} \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} r+1 & -\frac{r^2+2r}{v+y} \\ v+y & -r-1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & u+x+\frac{r^2-1}{v+y} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} r & -\frac{r^2-1}{v+y} \\ v+y & -r \end{bmatrix}.$$

Notice that $r^2 - 1 = 0$ if and only if $r \in \{\pm 1\}$ and $r^2 + 2r = 0$ if and only if $r \in \{0, 2\}$.

For $-X = -(v+y)(2r+1)(2\delta - 1)$ we obtain $t = (v+y)(1 + \frac{1}{1+4\delta})$ which is an integer if and only if $1 + 4\delta$ divides $v + y$. Again, this has to be continued with conditions on s.

Generally the relations $\alpha^2 + \alpha + uv = 0$ and $\beta^2 + xy = 0$, do not imply that $v+y$ divides any of r^2+1, r^2-1, $r^2+2r = (r+1)^2-1$ or $r^2+2r+2 = (r+1)^2+1$ (recall that $r = \alpha + \beta$), nor that $1 + 4\delta$ divides $5(v + y)$ (and so does not divide $v + y$).

Searching for a counterexample, we need integers $\alpha, \beta, u, v, x, y$ such that $\alpha^2 + \alpha + uv = 0 = \beta^2 + xy$, and $v+y$ does not divide any of the numbers: r^2+1, r^2-1, $(r+1)^2 - 1$ or $(r+1)^2 + 1$.

Further, $1 + 4\delta$ should not divide $5(v + y)$ and, moreover, to cover the trivial idempotents, we add two other conditions.

Since idempotents and units are clean in any ring, we must add: $\det \left[\begin{smallmatrix} r+1 & u+x \\ v+y & -r \end{smallmatrix}\right] \neq 0$ (this way the nil-clean matrix is not idempotent, nor nilpotent) and $\det \left[\begin{smallmatrix} r+1 & u+x \\ v+y & -r \end{smallmatrix}\right] \neq \pm 1$, (it is not a unit, and so nor unipotent), that is $\delta \notin \{0, \pm 1\}$.

Notice that if $r \in \{-2, -1, 0, 1\}$ then 0 appears among our two numbers ($r^2 - 1$, $(r+1)^2 - 1$) and the fraction is zero (i.e. an integer).

Since a matrix is nil-clean if and only if its transpose is nil-clean, we should have symmetric conditions on the corners $v + y$ and $u + x$, respectively. That is why, $u + x$ should not divide any of the numbers: $r^2 + 1$, $r^2 - 1$, $(r + 1)^2 - 1$ or $(r+1)^2 + 1$, and further, $1 + 4\delta$ should not divide $5(u + x)$.

Further, we exclude clean decompositions which use an idempotent of type $\left[\begin{smallmatrix} 0 & 0 \\ k & 1 \end{smallmatrix}\right]$. In this case the unit (supposed with $\det = -1$) should be $\left[\begin{smallmatrix} r+1 & u+x \\ (v+y)-k & -r-1 \end{smallmatrix}\right]$ and if its determinant equals -1 then $u+x$ divides r^2+r. Since idempotent, nilpotent, unit and so nil-clean matrices have the same property when transposed, to the conditions above we add $u + x$ and $v + y$ do not divide $r^2 + r$.

By inspection, one can see that there are no selections of $u + x$ and $v + y$ less than ± 7 and ± 9, at least for $r \in \{2, 3, \ldots, 10\}$, which satisfy all the above nondivisibilities.

Therefore $v + y = -7$, $u + x = 9$ is some kind of minimal selection. In order to keep numbers in the Pell equation as low as possible we choose $r = 2$ and so $\delta = -57$.

Indeed, our matrix verifies all *these exclusion conditions*: -7 and 9 do not divide any of $r^2 \pm 1 = 3, 5$, $(r+1)^2 \pm 1 = 8, 10$ nor $r^2 + r = 6$; $1 + 4\delta = -227$ (prime number) does not divide $5 \times (-7) = -35$ nor $5 \times 9 = 45$ and $\delta \notin \{0, \pm 1\}$.

Remark. We found this example in terms of r, δ, $u + x$ and $v + y$. It was not obvious how to come back to the nil-clean decomposition, that is, to α, β, u, v, x and y (indeed, this reduces to another elliptic Pell equation!). However, the following elementary argument showed more: there is only one solution, given by $(u, v) = (0, -6)$.

The system $\alpha + \beta = 2$, $u + x = 9$, $v + y = -7$, $\alpha^2 + \alpha + uv = 0 = \beta^2 + xy$ is equivalent to $(7u - 9v - 59)(7u - 9v - 54) + 25uv = 0$. Denote $t = 7u - 9v - 59$, hence $u = \frac{1}{7}(9v + t + 59)$. We obtain the equation

$$t(t + 5) + 25uv = 0. \tag{1}$$

Looking mod 5, it follows $t = 5k$, for some integer k. The equation simplifies to $k(k + 1) + uv = 0$. That is

$$k(k + 1) + \frac{1}{7}(9v + 5k + 59)v = 0.$$

Considering the last equation as a quadratic equation in k, we have

$$7k^2 + (5v + 7)k + 9v^2 + 59v = 0.$$

The discriminant of the last equation is

$$\Delta = (5v + 7)^2 - 28(9v^2 + 59v) = -227v^2 - 1582v + 49.$$

In order to have integer solutions for our last equation it is necessary $\Delta \geq 0$ and Δ to be a perfect square. The quadratic function $f(v) = -227v^2 - 1582v + 49$ has the symmetry axis of the equation $v_{\max} = -\frac{1582}{2 \cdot 227} < 0$, and $f(1) < 0$, hence there are no integers $v \geq 1$ such that $f(v) \geq 0$.

On the other hand, we have $f(-7) = 0$, giving $k = 2$, hence $t = 10$. Replacing in Eq. (1) we obtain $6 - 7u = 0$, equation with no integer solution. Moreover, we have $f(v) < 0$ for all $v < -7$.

From the above remark, it follows that all possible integer solutions for v are $-6, -5, -4, -3, -2, -1, 0$. Checking all these possibilities we obtain $f(-6) = 37^2$ and then $k = -1$. We get $t = -5$, and Eq. (1) becomes $-6u = 0$, hence $u = 0$.

6. A Related Question

Since both unit-regular and nil-clean rings are clean, a natural question is whether these two classes are somehow related. First $\mathbf{Z}_3$ (more generally, any domain with at least 3 elements) is a unit-regular ring which is not nil-clean, and, $\mathbf{Z}_4$ (more generally, any nil-clean ring with nontrivial Jacobson radical) is nil-clean but not unit-regular.

Finally, we give examples of nil-clean matrices in $\mathcal{M}_2(\mathbf{Z})$ which are not unit-regular, and unit-regular matrices which are not nil-clean.

Recall that the set of all the nontrivial nil-clean matrices in $\mathcal{M}_2(\mathbf{Z})$ is

$$\left\{ \begin{bmatrix} \alpha+\beta+1 & u+x \\ v+y & -\alpha-\beta \end{bmatrix} \middle| \alpha,\beta,u,v,x,y \in \mathbf{Z}, \alpha^2+\alpha+uv = 0 = \beta^2+xy \right\}$$

and that the only nonzero unit-regular matrices with a zero second row are $\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right]$, with (a,b) unimodular (i.e. a row whose entries generate the unit ideal, see [5]).

Hence $\left[\begin{smallmatrix} 2 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ is *unit-regular but not nil-clean* (nil-clean matrices have trace equal to 2,1 or 0; in the first case $\left[\begin{smallmatrix} 2 & 1 \\ 0 & 0 \end{smallmatrix}\right] - I_2$ is not nilpotent). Conversely, first notice that the nil-clean matrices with a zero second row are exactly the matrices $\left[\begin{smallmatrix} 1 & b \\ 0 & 0 \end{smallmatrix}\right]$, $b \in \mathbf{Z}$. Being idempotent, these are also unit-regular (so not suitable).

However, consider the nil-clean matrix (with our notations $\alpha = \beta = v = x = 0$, $u = 1$, $y = 2$) $A = \left[\begin{smallmatrix} 1 & 1 \\ 2 & 0 \end{smallmatrix}\right]$. Suppose A is unit-regular. Then, using an equivalent definition, $A = EU$ with $E = E^2$ and $U \in GL_2(\mathbf{Z})$. Since $\det A = -2 \neq \pm 1$, A is not a unit and so $E \neq I_2$. Hence $\det E = 0$ and from $\det A = \det E \cdot \det U$, we obtain a contradiction.

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