Research Article

Grigore Călugăreanu* and Horia F. Pop

On idempotent stable range 1 matrices

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Abstract: We characterize the idempotent stable range 1, 2×2 matrices over commutative rings and in particular the integral matrices with this property. Several special cases and examples complete the subject.

Keywords: idempotent stable range 1, clean, coprime integers, 2×2 matrix

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1 Introduction

The idempotent stable range 1 (isr1) for elements in a unital ring was introduced in [1] and further studied in [2].

Definition. An element a of a ring R is said to have (*left*) *stable range 1* (*sr1*) if for any $b \in R$, the equality $Ra + Rb = R$ implies that $a + rb$ is a unit for some $r \in R$. If r can be chosen to be an idempotent, we say that a has (*left*) *isr1*.

Actually, in [1], the definition of a (right) isr1 element was given requiring $a + br$ to be only left invertible, but it was immediately proved that this is equivalent to asking $a + br$ being a unit.

Since so far, left-right symmetry *for elements* with sr1 is an open question, we shall also consider that it is open for isr1 elements and discuss in the sequel about *left isr1 elements*. From the definition, we derive directly that a has *left isr1* iff for every $x, b \in R$ and $xa + b = 1$, there is an idempotent $e \in R$, called *unitizer* (as in [3]), such that $a + eb$ is a unit. Equivalently, for every $x \in R$, there is $e^2 = e \in R$ such that $a + e(xa - 1)$ is a unit. An element in a unital ring is called *clean* if it is a sum of an idempotent and a unit, and *strongly clean* if the idempotent and the unit commute.

Taking $e = 0$ shows that *units have* (not only sr1 but also) *isr1*. *Zero* has trivially isr1 (take $e = 1$).

Moreover, *isr1 elements are clean* (just take $x = 0$), but the converse fails (see the starting example of the next section).

In Section 2, we specialize the characterization given in [3], for sr1, 2×2 matrices over any commutative ring, to isr1 matrices and show that for such matrices, this notion is left-right symmetric. Next, we characterize the 2×2 integral isr1 matrices together with some special cases, including idempotents, nilpotents and matrices with zero second row. For the latter, "clean" and "isr1" turn out to be equivalent.

The last section is dedicated to examples and comments.

For any unital ring R, $U(R)$ denotes the set of all the units and $\mathbb{M}_2(R)$ denotes the corresponding matrix ring. By E_{ij} we denote the square matrix having all entries zero, excepting the (i, j) entry, which is 1.

*** Corresponding author: Grigore Călugăreanu,** Department of Mathematics, Babeş-Bolyai University, Cluj-Napoca, Romania, e-mail: calu@math.ubbcluj.ro
Horia F. Pop: Department of Computer Science, Babeş-Bolyai University, Cluj-Napoca, Romania, e-mail: hfpop@cs.ubbcluj.ro

2 2×2 isr1 matrices

First note that for any idempotent e, $2e$ is strongly clean: indeed, $2e = 1 + (2e - 1)$ and $2e - 1 = (2e - 1)^{-1} \in U(R)$.

Next we give an example of a *strongly clean element which does not have isr1*. Namely, we show (directly from definition) that $2E_{11}$ does not have isr1 in $\mathbb{M}_2(R)$, for any commutative ring R such that $2 \notin U(R)$ and $2R + 1 \not\subseteq U(R)$ (e.g., $R = \mathbb{Z}$). Suppose the contrary.

Let $a \in R$ such that $2a + 1 \notin U(R)$ and take $X = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$. By left isr(1), there is an idempotent (unitizer) E, such that $2E_{11} + E(2XE_{11} - I_2) \in U(\mathbb{M}_2(R))$. By computation $\det\left(2E_{11} + E\begin{bmatrix} 2a-1 & 0 \\ 0 & -1 \end{bmatrix}\right) \in U(R)$. Since the trivial idempotents $E = 0_2$ and $E = I_2$ are not suitable (as for the latter, the determinant is $-(2a + 1)$), we may assume $E = \begin{bmatrix} x & y \\ z & 1-x \end{bmatrix}$ with $x(1 - x) = yz$. Then $\det(2E_{11} + E(2XE_{11} - I_2)) = \det\begin{bmatrix} 2 + (2a-1)x & -y \\ (2a-1)z & x-1 \end{bmatrix} = 2(x - 1) \in U(R)$, a contradiction. Hence, this is an example of (*strongly*) *clean matrix that does not have isr1*.

The important properties that elements in the Ring theory may have, say, idempotent or nilpotent or unit, each separately, *are invariant under conjugations* but not each (separately) are invariant under equivalences. Namely, *idempotents and nilpotents are* (*separately*) *not invariant under equivalences*, but *units*, and more generally, *sr1 elements, are invariant under equivalences*. A simple example is $E_{11}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = E_{12}$, that is, an idempotent is equivalent to a nilpotent.

Therefore, as noted in [3], for the determination of sr1 matrices over elementary divisor rings, the *diagonal reduction* is useful, but *it is not, in the determination of* idempotents, or nilpotents or *isr1 elements*.

In particular, the proof of the *multiplicative closure for sr1 elements* (see [4]), does specialize to isr1.

Indeed, an example showing that the set of all the isr1 elements *is not* multiplicatively closed is given at the end of the paper.

First recall from [3], the following characterization.

Theorem 1. *Let R be a commutative ring and $A \in \mathbb{M}_2(R)$. Then A has left sr1 iff for any $X \in \mathbb{M}_2(R)$ there exists $Y \in \mathbb{M}_2(R)$ such that*

$$\det(Y)(\det(X)\det(A) - \mathrm{Tr}(XA) + 1) + \det(A(\mathrm{Tr}(XY) + 1)) - \mathrm{Tr}(A\,\mathrm{adj}(Y))$$

is a unit of R. Here $\mathrm{adj}(Y)$ *is the adjugate matrix.*

Notice that $\det(A(\mathrm{Tr}(XY) + 1)) = (\mathrm{Tr}(XY) + 1)^2 \det(A)$.

We obtain a *characterization for isr1 matrices* just adding the condition $Y^2 = Y$.

As this was done (see [3]) for sr1, 2×2 matrices, we obtain:

Corollary 2. *Let R be a commutative ring and $A \in \mathbb{M}_2(R)$. Then A has left isr1 iff A has right isr1.*

Proof. Using the properties of determinants, the properties of the trace and the commutativity of the base ring, it is readily seen that changing A, X, Y into transposes and reversing the order of the products does not change the condition in the previous theorem. ☐

We have also proved (see [3]) that a matrix $A \in \mathbb{M}_2(\mathbb{Z})$ has sr1 iff $\det A \in \{-1, 0, 1\}$. Since $\det(A) \in \{\pm 1\}$ yields precisely the units, in order to determine the isr1 integral 2×2 matrices, it remains to deal with (nonunits in) the case $\det(A) = 0$.

As our first main result we have the following characterization.

Theorem 3. *A noninvertible* 2×2 *integral matrix* A *has isr1 iff* $\det(A) = 0$, *the entries of* A *are (setwise) coprime and there exists a nontrivial idempotent* E *such that* $\mathrm{Tr}(AE) \in \{\pm 1\}$. *If* $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, *the last conditions are equivalent to the existence of integers* x, y, z *such that* $-a_{11}(1-x) + a_{12}z + a_{21}y - a_{22}x \in \{\pm 1\}$ *and* $x(1-x) = yz$.

Proof. Since sr1 integral matrices A have $\det(A) \in \{-1, 0, 1\}$ and we have excluded the units, $\det(A) = 0$ is necessary. The condition in the previous characterization becomes: for every $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, there is $Y = \begin{bmatrix} x & y \\ z & t \end{bmatrix} = Y^2$ such that

$$\det(Y)(-\mathrm{Tr}(XA) + 1) - \mathrm{Tr}(A\mathrm{adj}(Y)) \in \{\pm 1\}.$$

Over $\mathbb{Z}$, any (idempotent) unitizer is $Y = \begin{bmatrix} x & y \\ z & 1-x \end{bmatrix}$ with $x(1-x) = yz$ or else $Y \in \{0_2, I_2\}$.

The unitizer cannot be 0_2 (just by replacement) and it could be I_2 whenever $1 - \mathrm{Tr}(XA) - \mathrm{Tr}(A) \in \{\pm 1\}$. In the latter case, since for a given matrix A, $1 - \mathrm{Tr}(XA) - \mathrm{Tr}(A) \in \{\pm 1\}$ cannot hold for all $X \in \mathbb{M}_2(\mathbb{Z})$, for all the other X, if A is indeed isr1, Y must be a nontrivial idempotent.

Hence, we can assume $\det(Y) = 0$ and $\mathrm{Tr}(Y) = 1$, and so A has isr1 precisely when there are integers x, y, z such that $-\mathrm{Tr}(A\mathrm{adj}(Y)) \in \{\pm 1\}$, that is, $E = \mathrm{adj}(Y)$. Notice that $\det(Y) = \det(\mathrm{adj}(Y))$ and $\mathrm{Tr}(Y) = \mathrm{Tr}(\mathrm{adj}(Y))$, both are idempotent or not.

Therefore, (surprisingly) the unitizer is independent of X and the condition amounts to: a given matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ has isr1 iff there exist integers x, y, z such that $-a_{11}(1-x) + a_{12}z + a_{21}y - a_{22}x \in \{\pm 1\}$ and $x(1-x) = yz$.

Such integers (and the corresponding unitizer) exist iff the entries of A are (setwise) coprime and the coefficients in the linear combination give $\det(Y) = 0$ and $\mathrm{Tr}(Y) = 1$, as stated. □

In the sequel, we discuss several special cases.

Corollary 4. *A* 2×2 *integral matrix* A, *with three zero entries has isr1 iff the nonzero entry is* ± 1.

Proof. One way, just note that, for an integer n, $\{n, 0, 0, 0\}$ are (setwise) coprime iff $n \in \{\pm 1\}$. Conversely, it is easily checked that E_{22} is an (idempotent) unitizer for E_{11} and *vice versa*, $E_{11} + E_{21}$ is an (idempotent) unitizer for E_{12} and $E_{12} + E_{22}$ is an (idempotent) unitizer for E_{21}. □

The section begun by giving a direct (from definition) proof for *isr1* $(2E_{11}) \neq 1$. Now this also follows from the previous corollary.

Corollary 5. *Idempotent* 2×2 *integral matrices have isr1. Only the nilpotent matrices similar to* $\pm E_{12}$ *have isr1.*

Proof. The trivial idempotents are known to have isr1. Every nontrivial idempotent is similar (conjugate) to E_{11}, so has isr1. As for nilpotents, recall that every nilpotent matrix is similar to a multiple of E_{12}. Only those which are similar to $\pm E_{12}$ have isr1. Indeed, let $T = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$ with $x^2 + yz = 0$ be any nilpotent matrix and let $d = \gcd(x; y)$. Denote $x = dx_1$, $y = dy_1$ with $\gcd(x_1; y_1) = 1$. Then $d^2x_1^2 = -dy_1z$ and since $\gcd(x_1; y_1) = 1$ implies $\gcd(x_1^2; y_1) = 1$, it follows y_1 divides d. If $d = y_1y_2$ then T is similar to y_2E_{12}. □

Examples.

(1) $2E_{12}$ is not similar to any of $\pm E_{12}$. Indeed, if $(2E_{12})U = \pm UE_{12}$, for some U, then three entries of U vanish, so U cannot be a unit. Hence, $isr(2E_{12}) \neq 1$.

(2) In the case of nilpotents, $\begin{bmatrix} 6 & 3 \\ -12 & -6 \end{bmatrix}$ is similar to $3E_{12}$ and so does *not* have isr1, but $\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}$ is similar to E_{12} and so has isr1.

In what follows, we specialize our characterization Theorem 3 to the case of matrices with zero second row.

The case we deal with are matrices of form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, with nonzero coprime integers a, b (the case with three zeros was already settled). Note that we can suppose both a, b being positive.

Indeed, conjugation with $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ transforms $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ into $\begin{bmatrix} a & -b \\ 0 & 0 \end{bmatrix}$, and we pass from $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ to $\begin{bmatrix} -a & b \\ 0 & 0 \end{bmatrix}$ by rewriting the Bézout identity $ax + bz = 1$ as $(-a)(-x) + bz = 1$.

Such special matrices were studied with respect to cleanness in [5]. Two consecutive reductions were made there: from $a < b$ to $a > b$, and then from $a > b$ to $a \geq 2b$. We first show that these transformations can also be performed for isr1 matrices.

Lemma 6. *Suppose $a < b$ are coprime positive integers and $q \in \mathbb{Z}$. Then $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ has isr1 iff $\begin{bmatrix} a & b - qa \\ 0 & 0 \end{bmatrix}$ has isr1.*

Proof. One way is obvious (take $q = 0$). Conversely, suppose $ax + bz = 1$. Then $a(x + qz) + (b - qa)z = 1$ so $a, b - qa$ are also coprime. Both matrices (we denote these A and A') have zero determinant so, according to Theorem 3, if there is a nontrivial idempotent with $\mathrm{Tr}(AE) \in \{\pm 1\}$, we have to indicate a nontrivial idempotent E' such that $\mathrm{Tr}(A'E') \in \{\pm 1\}$. This amounts to complete the column $\begin{bmatrix} x + qz \\ z \end{bmatrix}$, to the right, up to a nontrivial idempotent. Set $E = \begin{bmatrix} x & y \\ z & 1 - x \end{bmatrix}$ with $x(1 - x) = yz$. Then $E' = \begin{bmatrix} x + qz & y + q - 2xq + q^2 z \\ z & 1 - (x + qz) \end{bmatrix}$ yields $\mathrm{Tr}(A'E') \in \{\pm 1\}$, as desired. □

As our second main result, we are now in position to prove.

Theorem 7. *For any coprime nonzero integers a, b and $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, the following conditions are equivalent:*
(*i*) *A has isr1;*
(*ii*) *A is clean.*

Proof. As already mentioned, we first show that there is no loss of generality in working with coprime (positive) integers $a \geq 2b$.

Indeed, if $0 < a < b$ and $b = qa + r$ is the division with quotient q and reminder r, we have $0 < r < a$ and, using the previous lemma, this is the passage from $a < b$ to $a > r$.

Next, suppose $b < a \leq 2b$. Then, using the previous lemma, we pass from $a > b$ to $a > b - a$ (here $b - a$ is negative), and finally we pass from $a > b$ to $a > a - b$. It just remains to note that $2b \geq a$ is equivalent to $a \geq 2(a - b)$ and we are done.

Since clean matrices A with $a \geq 2b$ were characterized in [5], by $a \equiv \pm 1 \pmod{b}$, it only remains to show that in this case A has also isr1. This follows from Theorem 3: if $a + bz = \pm 1$ we take $\mathrm{adj}(Y) = \begin{bmatrix} 1 & 0 \\ z & 0 \end{bmatrix}$, that is, $Y = \begin{bmatrix} 0 & 0 \\ -z & 1 \end{bmatrix}$ and so $\mathrm{Tr}(A\mathrm{adj}(Y)) = \pm 1$, as desired. □

3 Details on unitizers

We first recall briefly some details on the Bézout identity (over $\mathbb{Z}$, we discuss the solutions of a linear Diophantine equation with coprime coefficients).

If a, b are *coprime positive integers*, there exist integers x_0, z_0 such that $ax_0 + bz_0 = 1$. The other solutions of the equation $ax + bz = 1$ are $(x_0 + kb, z_0 - ka)$ for any integer k. Among these there exist precisely 2 *minimal* pairs (x, z), such that $|x| < b$, $|z| < a$. Clearly, for any solution, x and z have opposite signs. Moreover, one minimal solution has $x < 0$ and $z > 0$ and the other has $x > 0$ and $z < 0$.

The next simple result will be used.

Lemma 8. *Let (x_0, z_0) be a given solution of the equation $ax + bz = 1$ and $x = x_0 + kb$, $z = z_0 - ka$, the general solution. Then z divides $x - 1$ if there is an integer l such that $(ak - z_0)(al + b) = a - 1$.*

Proof. The divisibility amounts to $x_0 - 1 + kb = (z_0 - ka)l$ for some integer l. Multiplying by a, decomposing, and using $ax_0 + bz_0 = 1$ gives the equality in the statement. □

Since $a - 1$ has finitely many divisors, the equality cannot be satisfied for every integer k, so even if z_0 divides $x_0 - 1$, not all z divide $x - 1$.

Clearly, a necessary condition for this divisibility is that $al + b$ divides $a - 1$ for some integer l.

Also note that we actually consider $ax + bz = \pm 1$, so the solutions (x, z) of $ax + bz = 1$, but also $(-x, -z)$.

Next, in order to avoid the double reduction in Theorem 7 and to have some direct proofs (of isr1) for matrices with zero second column, in the next result we provide unitizers in *all* the possible cases.

Theorem 9. *Suppose a, b are coprime positive integers and let x, z be any solution to $ax + bz = 1$. The matrix $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ with $a < b$ has isr1 iff z divides $1 - x$ or $1 + x$. If $a > b$, A has isr1 iff $x = 1$ or else $|z| = |1 \pm x|$.*

Proof. Suppose a and b are coprime positive integers and let x, z be any solution to $ax + bz = 1$. As already seen before, for the conditions $\mathrm{Tr}(\mathrm{adj}(Y)) = \pm 1$, we just have to complete the column $\begin{bmatrix} x \\ z \end{bmatrix}$ to the right, up to a nontrivial idempotent $\begin{bmatrix} x & y \\ z & 1 - x \end{bmatrix}$ together with $x(1 - x) = yz$. Since x and z are coprime, this amounts to some divisibilities.

(A) Suppose $0 < a < b$. Here $|x| > |z|$, these have opposite signs and z must divide $x - 1$ or $x + 1$ (as noted, the solution $(-x, -z)$ is also suitable).

(1) For $ax + bz = 1$, suppose z divides $x - 1$, that is, $x - 1 = kz$, for some integer k. We take $\mathrm{adj}(Y) = \begin{bmatrix} x & -kx \\ z & -kz \end{bmatrix}$ for which $\det(\mathrm{adj}(Y)) = 0$ and $\mathrm{Tr}(\mathrm{adj}(Y)) = 1$ so $\mathrm{adj}(Y)$ is idempotent. So is Y and since $A\mathrm{adj}(Y) = \begin{bmatrix} 1 & -k \\ 0 & 0 \end{bmatrix}$, $\mathrm{Tr}(A\mathrm{adj}(Y)) = 1$ follows and we are done.

(2) For $ax + bz = 1$, suppose z divides $x + 1$, that is, $x + 1 = kz$, for some integer k. We take $\mathrm{adj}(Y) = \begin{bmatrix} -x & kx \\ -z & kz \end{bmatrix}$ for which $\det(\mathrm{adj}(Y)) = 0$ and $\mathrm{Tr}(\mathrm{adj}(Y)) = 1$ so $\mathrm{adj}(Y)$ is idempotent. So is Y and since $A\mathrm{adj}(Y) = \begin{bmatrix} -1 & k \\ 0 & 0 \end{bmatrix}$, $\mathrm{Tr}(A\mathrm{adj}(Y)) = -1$ follows, as desired.

(B) If $a > b > 0$ are coprime integers and $ax + bz = 1$, then $|x| < |z|$ and these have opposite signs ($x > 0 > z$ or else $x < 0 < z$). Here $x \in \{\pm 1\}$ or $|z| = |1 \pm x|$.

(1) If $x = 1$ we have a unitizer indicated in the proof of Theorem 7. The case $x = -1$ is similar.

(2) If $|z| = |1 \pm x|$, we find unitizers as in the case **A** above. □

Related to the divisibilities above, we give the following:

Examples.

(1) For $(a, b) = (8, 13)$, the minimal pairs are $(5, -3)$ and $(-8, 5)$. For none z divides $x - 1$, but for $(5, -3)$, -3 divides $5 + 1$.

(2) For $(15, 23)$, the minimal pairs are $(-3, 2)$ and $(20, -13)$. Here 2 divides $-3 - 1$.

(3) For $(5, 7)$, the minimal pairs are $(3, -2)$ and $(-4, 3)$. Here -2 divides $3 - 1$ and also 3 divides $-4 + 1$.
(4) For $(5, 9)$, the minimal pairs are $(2, -1)$ and $(-7, 4)$. Here -1 divides 2 ± 1 and also 4 divides $-7 - 1$.

Nonexamples.
(1) For $(12, 17)$, the minimal pairs are $(10, -7)$ and $(-7, 5)$.
(2) For $(12, 19)$, the minimal pairs are $(8, -5)$ and $(-11, 7)$.
(3) For $(12, 31)$, the minimal pairs are $(13, -5)$ and $(-18, 7)$.
(4) For $(13, 18)$, the minimal pairs are $(7, -5)$ and $(-11, 8)$.
(5) For $(51, 71)$, the minimal pairs are $(-32, 23)$ and $(39, -28)$.

Remark. While for *examples*, indicating a pair (minimal or not) is sufficient in order to construct an idempotent unitizer, and so to check the isr1 property, for the *nonexamples* (as the referee pointed out) a proof is necessary in order to show that if the conditions in the previous theorem are *not* fulfilled for a given pair (e.g., a minimal pair), these are *not* fulfilled for any other solution of $ax + bz = 1$. This is clear in the **B** case and not hard to check in the **A** case.

Recall that if (x_0, z_0) is a (minimal) solution for $ax + bz = 1$, then the other solutions are given by $x = x_0 + kb$, $z = z_0 - ka$. Of course, it would suffice to show that if z_0 does not divide $x_0 - 1$ then every z does not divide the corresponding $x - 1$.

Unfortunately, this is not true in general as it shows the following.

Example. For $(a, b) = (2, 5)$, a solution pair is $(-7, 3)$ for which 3 does not divide $-8 = -7 - 1$. The general solution $x = -7 + 5k$, $z = 3 - 2k$ gives for $k = 1$ the (minimal) solution $(-2, 1)$ for which 1 divides $-3 = -2 - 1$.

Therefore, another type of verification is necessary for the nonexamples, starting with a given pair (e.g., a minimal pair). The implication "z_0 divides $x_0 - 1$ then z divides $x - 1$" must be verified for each nonexample, separately, and this amounts to solve a quadratic Diophantine equation! As seen in Lemma 8, this equation is

$$(ak - z_0)(al + b) = a - 1$$

with unknowns k, l. Especially when $a - 1$ is a prime, it sometimes suffices to check that $al + b$ does not divide $a - 1$. This can be easily done for Examples (1)–(3).

Next, we reconsider Example 4, from the aforementioned nonexamples, and give all details.

For $13x + 18z = 1$ the minimal pairs are $(x_0, z_0) \in \{(7, -5), (-11, 8)\}$, for which clearly z_0 does not divide $x_0 \pm 1$.

It suffices to discuss one minimal pair, say the first pair, for which the general solution is $x = 7 + 18k$, $z = -5 - 13k$. Would z divide $x - 1$, then $6 + 18k = -(5 + 13k)l$ for some integer l.

As seen above, this quadratic Diophantine equation in the unknowns k, l can be written $(13k + 5)(13l + 18) = 12$.

That this equation has no solutions can be shown browsing the pairs of two integers whose product is 12, or using any software online (e.g., [6]).

If z would divide $x + 1$, similarly we reach the equation $8 + 18k = -(5 + 13k)l$, also with no solutions.

Note that we used the minimal pairs just to keep the numbers low.

As applications of our results, here are some more

Examples.

(1) We can show that $A = \begin{bmatrix} 5 & 12 \\ 0 & 0 \end{bmatrix}$ *has* isr1, in two different ways.
  (i) The minimal solutions (x, z) of $ax + bz = 1$ are $(5, -2)$, $(-7, 3)$ and -2 divides $5 - 1$. Moreover, 3 divides $-7 + 1$. Therefore, we indicate a (nontrivial idempotent) unitizer as in **A**, above: $Y = \begin{bmatrix} -4 & -10 \\ 2 & 5 \end{bmatrix}$.
  (ii) As described in the previous section, we pass $(12 = 2 \cdot 5 + 2)$ from $(5, 12)$ to $(5, 2)$ for which $5 \equiv 1 \pmod 2$, so A is clean and isr1 (by Theorem 7). Actually $\begin{bmatrix} 5 & 12 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -4 & -10 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 9 & 22 \\ -2 & -5 \end{bmatrix}$ is its (uniquely) clean decomposition.

(2) However, we can show that $B = \begin{bmatrix} 12 & 5 \\ 0 & 0 \end{bmatrix}$ *has not* isr1 in three different ways.

(i) Since $12 > 5$ and $12 \geq 2 \cdot 5$ we can use Theorem 7: $12 \not\equiv \pm 1 \pmod 5$ so B is not clean (and so nor isr1).

(ii) For $X = 0_2$, if a unitizer Y exists, we would have $A - Y \in U(\mathbb{M}_2(\mathbb{Z}))$, that is, A would be clean. We can show that this fails (e.g., see [7]) solving the Diophantine equations $5x^2 - 12xy + 5x \mp y = 0$ together with $12x + 5z = \pm 1$ (the only solutions are $(0, 0)$, $(-1, 0)$, none verifies both equations), or else, using [5], where this matrix and others are given examples of (unit-regular) matrices which are not clean.

(iii) We are in the **B** case above. The minimal pairs for $(12, 5)$ are $(x, z) \in \{(3, -7), (-2, 5)\}$. For none $x \in \{\pm 1\}$ nor $|z| = |1 \pm x|$, and this is sufficient according to the previous remark.

Remark. The case when the first row is zero (or some column is zero) reduces to the previous discussed case.

By conjugation with $U = E_{12} + E_{21}$, we check that $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ is similar to $A' = \begin{bmatrix} 0 & 0 \\ b & a \end{bmatrix}$, and $B = \begin{bmatrix} b & a \\ 0 & 0 \end{bmatrix}$ is similar to $B' = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$. Both A, A' may have isr1 but B, B' may not have isr1 $\left(\text{see previous example: } \begin{bmatrix} 5 & 12 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 12 & 5 \\ 0 & 0 \end{bmatrix} \right)$. As for zero columns we just use the transpose.

In closing, we provide an example of two isr1 integral matrices whose product does *not* have isr1.

Example. Take $A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$. Since (by Theorem 3), $\operatorname{adj}(Y) = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ gives a suitable (idempotent) unitizer, A has isr1. However, $A^2 = \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix}$ does not have isr1, because its entries are not (setwise) coprime.

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