ADDITIVE GROUPS OF RINGS WITH IDENTITY

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Abstract. A ring with identity exists on a torsion Abelian group exactly when the group is bounded. The additive groups of torsion-free rings with identity are studied. Results are also given for not reduced splitting mixed rings with identity. The Abelian groups G such that, excepting the zero-ring, every ring on G has identity are also determined.

1. Introduction

In the sequel, all the groups we consider are nonzero Abelian, and all the rings are nonzero and associative. As customarily, for a ring R, we denote the additive (Abelian) group by R^+, and for an Abelian group G, we say that R is a ring on G if $R^+ = G$. Hereafter, a group will be called an *identity-group* (identity for short) if there exists an associative ring with identity on G and *strongly identity-group* (S-identity for short), if it is identity and, excepting the zero-ring, all associative rings on G have identity. A group G is called *nil* if the only ring on G is the zero-ring. Clearly nil groups are not identity nor S-identity.

Since unital rings embed in the endomorphism rings of their additive groups it follows that identity-groups are isomorphic to additive groups of unital subrings of endomorphism groups. As a special case, endo-groups (and among these, additive groups of so-called E-rings) are identity-groups. Here an Abelian group G is called an *endo-group* if there is a ring R over G such that $R \cong \mathrm{End}(G)$, a ring isomorphism. However, a simple comparison (see **4.6.7-4.6.11**, [1]) shows that endo-group is far more restrictive than identity-group (e.g. any finitely generated group is identity, but only cyclic groups are endo-groups).

It was known from long time that *a torsion group is the additive group of a ring with identity if and only if it is bounded*, i.e., a bounded direct sum of finite (co)cyclic groups. Consequently, it is not hard to show that *the only S-identity torsion groups are the simple groups* $\mathbf{Z}(p)$, for any prime number p (for a proof, see next Section). Moreover, the only torsion-free S-identity group is $\mathbf{Q}$, the full rational group, and there are no mixed S-identity groups.

Since there are no results on torsion-free or mixed identity-groups, this is what we investigate in this note. In the torsion-free case we obtain results when the typeset $\mathbf{T}(G)$ contains a minimum element which is idempotent and in the mixed case significant results are found in the case of splitting mixed groups which are not reduced.

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In the general case, the problem seems to be difficult (as many problems related to torsion-free or mixed rings are). Thus, this note also intends to open a possible new area of research.

For definitions, notations and results on Abelian groups we refer to L. Fuchs [2]. The largest divisible subgroup of an Abelian group G will be called the *divisible part* of G, and a direct complement of the torsion part will be called a *torsion-free part*. For results on additive groups of rings we refer to S. Feigelstock [1]. The left annihilator of a ring R is $\mathrm{ann}_l(R) = \{a \in R | aR = 0\}$. Similarly, the right and the annihilator of a ring are defined. Obviously, any ring with identity must have zero left (or right, or two-sided) annihilator. Since in any ring, disjoint ideals annihilate each other, in the additive group of a ring, different primary components annihilate each other, and any torsion-free ideal annihilates the torsion part. For a subgroup H of R^+, (H) denotes the ideal generated by H in R.

2. Strongly identity groups

Any ring multiplication may be extended from a direct summand, by taking the ring direct sum with the zero-ring on a direct complement (also called a *trivial* extension). Since a finite direct (product) sum of rings has identity if and only if each component has identity (the finite decomposition of 1 into central idempotents), the ring obtained by any trivial extension has no identity. Hence *S-identity groups are indecomposable*.

The strongly identity groups are rare phenomenon.

Theorem 1. *A group is S-identity if and only if it is isomorphic to* $\mathbf{Z}(p)$*, for a prime number* p*, or isomorphic to* $\mathbf{Q}$.

Proof. Since there are no mixed indecomposable groups, we expect to find only torsion or torsion-free S-identity groups. Let G be an identity group with multiplication denoted $x \cdot y$ for $x, y \in G$. For a prime p, consider the multiplication (associative and both left and right distributive together with $\cdot$) $x \circ y = p(x \cdot y)$. Since this ring must be the zero ring or an unital ring, $pG = 0$ or $pG = G$ for every prime p. In the first case, since G is indecomposable (and so cocyclic) $G \cong \mathbf{Z}(p)$. In the second case, G is indecomposable divisible and so $G \cong \mathbf{Z}(p^\infty)$ (impossible, because not identity) or $G \cong \mathbf{Q}$. Conversely, these two groups are S-identity. Indeed, for every $k = 1, ..., p-1$ the multiplication by k (on $\mathbf{Z}(p)$) has identity: it is just the multiplicative inverse $\overline{k}^{-1}$ in the field $\mathbf{Z}(p)$. As for $\mathbf{Q}$, recall that multiplications on $\mathbf{Q}$ are determined by nonzero squares a^2 of rational numbers ([2], p. 291). That is, for nonzero elements $c, d \in \mathbf{Q}$ there are nonzero rationals r, s such that $c = ra$ and $d = sa$, and $c \cdot d = (rs)a^2$. Since $a \neq 0$, $1 = a^{-1}a$ is the identity for this (commutative, associative and without zero divisors) multiplication. □

3. Torsion-free identity rings

It is readily checked that free groups and divisible torsion-free groups are identity (more: a torsion-free group is a field-group if and only if it is divisible; see [1]). Hence, if $G = D(G) \oplus H$, is a decomposition with the divisible part $D(G)$ and a reduced direct complement H, using the ring direct product, it follows at once that G is an identity-group whenever H is so.

For easy reference we mention here ([2], **123.2**) a generalization of the Dorroh ring extension.

Proposition 2. *Let R be an A-algebra, with A a commutative ring with identity. Then the ring R_A with $R_A^+ = A^+ \oplus R^+$ and multiplication defined by $(a_1, r_1) \cdot (a_2, r_2) = (a_1 \cdot a_2, a_1r_2 + a_2r_1 + r_1r_2)$ for all $a_1, a_2 \in A$, $r_1, r_2 \in R$, is a ring with identity. The map $R \longrightarrow R_A$ via $r \longmapsto (0, r)$ is an embedding of R in R_A as an ideal, and $R_A/R \cong A$.*

Corollary 3. *For any group G, the direct product (sum) $\mathbf{Z} \times G$ is an identity-group.*

Proof. It suffices to take the trivial multiplication on G. □

In the torsion free case we have a converse for this corollary:

Proposition 4. *Let $X = \{e, g_i | i \in I\}$ be a maximal independent system in G. The following are equivalent:*

(1) the partial operation defined by $e \cdot e = e$, $e \cdot g_i = g_i \cdot e = g_i$, $g_i \cdot g_k = 0$ extends to a (ring) multiplication with identity e on G;
(2) the characteristic $\chi(e)$ is idempotent and minimum, and $\langle e \rangle_*$ is a direct summand of G.

Proof. (1)⇒(2) (i) In order to define $e \cdot e = e$, since $\chi(ab) \geq \chi(a)\chi(b)$, we need $\chi(e) \geq \chi(e)^2$, and so (by definition of the characteristics product) $\chi(e) = \chi(e)^2$, i.e., e has idempotent characteristics. Further, since we need, $e \cdot a = a \cdot e = a$ for every $a \in G$, $\chi(a) \geq \chi(a)\chi(e) = \chi(e)\chi(a) \geq \chi(e)$ shows that e has minimum characteristic.

(ii) Denote by $N = \langle g_i | i \in I \rangle_*$. Then $N \cdot N = 0$. Since $\langle e \rangle_* \cap N = 0$, it remains to prove that $G = \langle e \rangle_* + N$.

Let $x \in G$ be arbitrary. Since every element depends on a maximal independent set, there are nonnegative integers n, k such that $nx = ke + y$, with $y \in N$. If $d = (n, k)$, then d divides y and $(1/d)y \in N$ (because N is pure). Thus, dividing $nx = ke + y$ by d, we can suppose $(n, k) = 1$.

Multiplying the dependence relation by y yields $nx \cdot y = ky$.

If $uk + vn = 1$, then $y = uky + vny = unx \cdot y + vny$, and so n divides y. Since $(n, k) = 1$, and n divides y, it also divides the identity e. Thus, G itself is n-divisible (indeed, for every $g \in G$, $g = g \cdot e$ and $n|e$ imply $n|g$) and $x = (k/n)e + (1/n)y$, so $x \in \langle e \rangle_* + N$ (again $(1/n)y \in N$, because N is pure).

(2)⇒(1) Note that by (i) and (ii) it follows that $N = \langle g_i | i \in I \rangle_*$ is a direct complement for $\langle e \rangle_*$. Moreover, since the type of e is idempotent then $\langle e \rangle_*$ has a natural multiplication such that it becomes a ring isomorphic to a unital subring $P^{-\infty}\mathbf{Z}$ of $\mathbf{Q}$ generated (as subring) by all $\frac{1}{p}$ with $p \in P$, where P is the set of all primes p with the property that $\langle e \rangle_*$ is p-divisible. Since N is a $P^{-\infty}\mathbf{Z}$-module, we can apply the previous proposition to obtain the conclusion. □

Remark. In the unital ring on G that occurs by this construction, $1 \in P^{-1}\mathbf{Z}$ is the identity. Denoting it by $e \in G$, the multiplication (with identity e) on G is given by linearly extending (over $P^{-1}\mathbf{Z}$) $e \cdot e = e$, $e \cdot h = h \cdot e = h$, $h \cdot h_1 = 0$ $(h, h_1 \in H)$. Observe that the characteristic of $\chi(e)$ is idempotent and minimum in G.

This way, torsion-free identity-groups which admit a Dorroh-like ring multiplication are characterized.

Remark. In a torsion-free group, the subgroup purely generated by an element of idempotent and minimum characteristic might not be a direct summand. For instance, consider two rank 1 nil groups H, K with types $\mathbf{t}(H) = (0,1,0,1,...)$ and $\mathbf{t}(K) = (1,0,1,0,...)$. Then the direct sum $H \oplus K$ contains an element $h+k$ of (minimum and idempotent) characteristics $(0,0,...)$, with $\chi(h) = (0,1,0,...)$ and $\chi(k) = (1,0,1,...)$. But the subgroup purely generated by $h+k$ is not a direct summand.

For the case of mixed groups (by a mixed group we mean a genuine mixed group, i.e. $0 \neq T(G) \neq G$), since *there exist no nil mixed groups* (Szele, **120.3** [2]), when finding identity-groups, no mixed groups have to be excepted.

Recall (**4.6.3**, [1]): *let R be a ring with trivial left annihilator and $R = A + B$ for subsets A, B of R. If $A^2 = B^2 = 0$ then $R = 0$.*

Hence, *a (direct) sum of two (nonzero) nil groups is not identity* and so, *a mixed group with divisible torsion part and nil torsion-free complement is not identity.* But nil torsion-free groups are far of being known, so this covers only a few known situations.

We first settle the case when the torsion-free part is divisible.

Proposition 5. *A mixed group with divisible torsion-free part is identity if and only if the torsion part is bounded.*

Proof. Such groups are splitting and so have the form $G = T(G) \oplus (\bigoplus \mathbf{Q})$ with torsion part $T(G)$. If R is a ring on G, it is proved in [1] (**4.3.15**), that any ring with trivial annihilator on the group direct sum above is also a ring direct sum on the components (i.e., it is *fissible*). Since identity rings have trivial annihilator, and ring direct sums have identity if and only if the components have identities, $T(G)$ must be bounded. Conversely, $D(G) = \bigoplus \mathbf{Q}$ is known to be identity and so, if $T(G)$ is bounded, G is identity. □

The *second* case we discuss is when the torsion part is not reduced.

Proposition 6. *If the torsion part is not reduced, i.e., $G = DT(G) \oplus H$ with (maximal) torsion divisible $DT(G)$, and R is a ring with identity on G, then (a) for any relevant prime relative to $DT(G)$, H is not p-divisible and (b) $h_p(1) = 0$ (in both H or G).*

Proof. (a) As intersection of two ideals (the torsion part and the divisible part are fully invariant subgroups), $DT(G)$ is also an ideal which, being torsion divisible, must be nil (ideal), i.e., $t \cdot t' = t' \cdot t = 0$ for every $t, t' \in DT(G)$.

If H is p-divisible then $G = \mathbf{Z}(p^\infty) \oplus K$ and $\mathbf{Z}(p^\infty)$ annihilates R and so G is not identity group. Indeed, let $x \in R$, $a \in \mathbf{Z}(p^\infty)$ with $\mathrm{ord}(a) = p^k$. Since G is p-divisible, there is $y \in G$ such that $x = p^k y$ and it is readily checked that $a \cdot x = x \cdot a = 0$.

(b) For any relevant prime p (relative to $DT(G)$) we show that $1 \notin pG$. Indeed, otherwise $1 \in pG$ implies $pG = G$, the left member being ideal in R, which is impossible: $pG = p(T(G) \oplus H) = pT(G) \oplus pH = T(G) \oplus pH < G$. □

The *third* case is with a divisible torsion part, again splitting, so let R be a ring on $G = T(G) \oplus V$, with reduced torsion-free V. Here we must except the (already discussed) identity case when V has a cyclic direct summand.

If for a relevant prime p, V is p-divisible, then $\mathbf{Z}(p^\infty)$ annihilates all R and so G *is not identity.*

The remaining case is when $pV \neq V$ for all relevant primes. Some groups of this type are considered in [1](p. 8).

Suppose $G = H \oplus \mathbf{Z}(p^{\infty})$ with torsion-free not p-divisible H. Choose $b \in H$ with $h_p(b) = 0$. For any positive integer n, choose $a_n \in \mathbf{Z}(p^{\infty})$ with $\mathrm{ord}(a_n) = p^n$. The map $b \otimes b \longmapsto a_n$ can be extended to an epimorphism $H \otimes H \longrightarrow \mathbf{Z}(p^n)$.

Therefore a ring R with $R^+ = G$ can be constructed so that $(H) \cap \mathbf{Z}(p^{\infty})$ is an arbitrary proper subgroup of $\mathbf{Z}(p^{\infty})$.

However, these are not unital rings. Indeed, denote by R a ring constructed as above. Then

$$R/(H) = \frac{(H) + \mathbf{Z}(p^{\infty})}{(H)} \cong \frac{\mathbf{Z}(p^{\infty})}{\mathbf{Z}(p^n)} \cong \mathbf{Z}(p^{\infty})$$

is a divisible p-group, which (not being bounded) is not identity. Hence so is G.

Finally, note that a consequence of Proposition 2 also gives mixed identity-groups, using the Dorroh-like construction. That is

Corollary 7. *Let P be a set of prime numbers and G a p-divisible group with zero p-components for all $p \in P$. Then the direct sum $(P^{-1}\mathbf{Z})^+ \oplus G$ is an identity-group with respect to the Dorroh-like multiplication defined by $(q_1, x_1) \cdot (q_2, x_2) = (q_1.q_2, q_1x_2 + q_2x_1)$ for all $q_1, q_2 \in P^{-1}\mathbf{Z}$, $x_1, x_2 \in G$.*

Here, for a set of prime numbers P, $\varnothing \subseteq P \subseteq \mathbf{P}$, $P^{-1}\mathbf{Z}$ denotes the unital subring of $\mathbf{Q}$ generated (as subring) by all $\frac{1}{p}$ with $p \in P$. Obviously $(P^{-1}\mathbf{Z})^+$ is p-divisible for all $p \in P$.

In closing this paper here are two results also related to our subject

Proposition 8. *A group has only identity-subgroups if and only if it is a direct sum of a bounded group and a torsion-free identity-group which has only identity-subgroups.*

Proof. Suppose G has only identity-subgroups. Since its torsion part must be identity, $T(G)$ (together with its subgroups) is bounded and so (Baer-Fomin) G is splitting: $G = T(G) \oplus V$. Here V, but also all its subgroups, must be identity and so, must have an element of minimum and idempotent type. The converse is obvious. □

Despite the fact that, having only identity-subgroups seems a strong condition on torsion-free groups, we were not able to give a useful characterization for such groups. Since the subgroups purely generated by elements must have idempotent type (otherwise these are nil and so not identity) such groups have only idempotent types in their typeset (more, these types have only finitely many nonzero components). Among finite rank torsion-free Butler groups (a Butler group is a homomorphic image of a completely decomposable group) the quotient divisible groups are exactly those which have only idempotent types (see [3]), **8.6**).

Proposition 9. *A group has only identity quotient groups if and only if it is a direct sum of a bounded group and a finite rank free group.*

Proof. Suppose G has only identity quotient groups. If G has infinite rank, there is an epimorphism $G \longrightarrow \mathbf{Q}/\mathbf{Z}$, a contradiction.

Thus G has finite rank. If we take a torsion-free quotient of rank 1, this has idempotent type. If this is not free, again we can find a torsion divisible quotient, a contradiction. Therefore, every rank 1 torsion-free quotient is free and so G (having finite rank) has the decomposition we claimed: $G = T \oplus F$ with a finite rank free F.

Finally, if some p-component is not bounded, again we can find an epimorphism to a torsion divisible group, a contradiction.

Since bounded groups and free groups are identity-groups, the converse follows. □

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