Sections and retractions in **Grp**.

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21 January 2011

Abstract

Sections and retractions in the category of groups are not discussed in Category Theory textbooks. However, in this short note we show that these can be characterized in an elementary way, pointing out a nice duality.

1 Introduction

In [1], discussing sections and retractions, Adamek, Herrlich and Strecker, give characterizations of sections and retractions in the categories **Set**, **Vec**, **Top** and **Ab** but the category of groups, **Grp**, is not mentioned.

Since the author also doesn't know any bibligraphic reference about sections and retractions in **Grp**, in the short note we fill this (very likely) gap. Moreover, our results point out a nice duality.

2 Sections

Proposition 1 *[1]In **Ab** a morphism $f : A \longrightarrow B$ is a section if and only if it is an injective function and $f(A)$ is a direct summand of B.*

It is a little bit harder to characterize sections in **Grp**. We first need the following

Definition 2 *Let H be a subgroup of G. A complement of H in G is a subgroup K of G such that $HK = G$ and $|H \cap K| = 1$.*

Equivalently, a complement is a *transversal* of H (a set containing one representative from each coset of H) that happens to be a group.

Remark 3 *Let K be a complement of H in a group G. Then every element g in G has a unique representation as $g = hk$ with $h \in H$ and $k \in K$.*

Indeed, if $g = hk = h_1k_1$ then $h_1^{-1}h = k_1k^{-1} \in H \cap K$ and so $h_1^{-1}h = 1 = k_1k^{-1}$ and $h = h_1$, $k = k_1$.

Lemma 4 *Let $i : H \longrightarrow G$ be the inclusion of a subgroup H in a group G. A group morphism $p : G \longrightarrow H$ with $p \circ i = 1_H$ exists if and only if H has a normal complement.*

Proof. Let K be a normal complement for H in G. Using the previous Remark, define $p : G \longrightarrow H$ by $p(g) = h$ if $g = hk$. Let $g = hk$ and $g_1 = h_1k_1$ be elements in G. Then $p(g)p(g_1) = hh_1 = p(gg_1)$ because $gg_1 = hkh_1k_1 = hh_1k'k_1$ for a suitable $k' \in K$ (K is normal in G). Since obviously $h = h.1$, we obtain $p(h) = h$ for every $h \in H$ and finally $p \circ i = 1_H$. Conversely, suppose the inclusion $i : H \longrightarrow G$ has a left inverse $p : G \longrightarrow H$. Denote by $K = \ker p$. Since $p(h) = h$ for every $h \in H$ and $p(k) = 1$ for every $k \in K$, we obtain $K \cap H = 1$. Further, for every $g \in G$ notice that $p(p(g^{-1}).g) = p(p(g^{-1})).p(g) \overset{*}{=} p(g^{-1})p(g) = 1$ and so $p(g^{-1}).g \in \ker p = K$ (here $p(g^{-1}) \in H$ justifies *). Now $g = p(g)[p(g^{-1}).g] \in HK$ shows that K is a complement of H in G. Finally, as a (group theoretic) kernel, K is a normal subgroup in G. $\blacksquare$

Having recalled all this, the proof of the following characterization is suggested by the previous proof (in **Ab**).

Proposition 5 *In **Grp** a morphism $f : A \longrightarrow B$ is a section if and only if it is an injective function and $f(A)$ has a normal complement in B.*

Proof. Again, notice that if f is an injective group morphism, then $\widetilde{f} : A \longrightarrow f(A)$ is a group isomorphism. Suppose K is a normal complement of $f(A)$ in B, that is $B = f(A)K$ and $|f(A) \cap K| = 1$. According to the previous Lemma, there exists a projection $p : B \longrightarrow f(A)$, i.e., a group epimorphism such that $p \circ i = 1_{f(A)}$, with $i : f(A) \longrightarrow B$ the inclusion. Then $g = (\widetilde{f})^{-1} \circ p : B \longrightarrow A$ is a left inverse for f. Conversely, if there is a group morphism $g : B \longrightarrow A$ such that $g \circ f = 1_A$ consider the idempotent endomorphism $f \circ g : B \longrightarrow B$. Since g is surjective and f is injective, $\mathrm{im}(f \circ g) = \mathrm{im}(f) = f(A)$ and $\ker(f \circ g) = \ker g = K$. We show that K is a (clearly normal) complement of $f(A)$ in B. Since $f \circ g$ is idempotent, $\mathrm{im}(f \circ g) \cap \ker(f \circ g) = f(A) \cap K = 1$. Finally, $B = f(A)K$ is obtained as in the proof of the above Lemma. $\blacksquare$

Remark 6 *Not every monomorphism in **Grp** is a section.*

According to the above Proposition, our task is easier. We need a subgroup which is an image of a group morphism, but has no (normal) complement. Since normality is automatic in abelian groups, a subgroup H of an abelian group G has a normal complement K if and only if $G = H \times K$, a direct product.

Take $H = 2\mathbf{Z}_4$ in $G = \mathbf{Z}_4$ and the inclusion. H has no complement in G.

3 Retractions

Remark 7 *Not every group epimorphism is a retraction.*

Indeed, consider the natural (projection) map $p : \mathbf{Z} \longrightarrow \mathbf{Z}_2$ $(p(2n) = \overline{0}$, $p(2n+1) = \overline{1})$ which is clearly surjective and so group epimorphism. Since $\mathrm{Hom}_{\mathbf{Grp}}(\mathbf{Z}_2 \longrightarrow \mathbf{Z})$ consists only of the zero group morphism, p has no left inverse group morphism. Of course, this example can be also used in the subcategory **Ab**.

Proposition 8 *[1]In* ***Ab*** *$f : A \longrightarrow B$ is a retraction if and only if there is an abelian group C such that $A \xrightarrow{f} B = A \xrightarrow{h} B \times C \xrightarrow{p_B} B$, where h is an isomorphism and p_B is the projection.*

The characterization of a retraction in **Grp** is similar (and suggested by the previous Proposition) and points out a nice duality

Proposition 9 *In* ***Grp*** *$f : A \longrightarrow B$ is a retraction if and only if f is a surjective function and* $\ker f$ *is a (normal) complement of a subgroup H in A.*

Proof. Let $\ker f$ be a (normal) complement of a subgroup H in A. Then $H \cap \ker f = 1$, $A = H \cdot \ker f$ and $g : B \longrightarrow A$ given by the composition (one uses here first and second Noether isomorphism theorems) $B = f(A) \cong A/\ker f = (H \cdot \ker f)/\ker f \cong H/(H \cap \ker f) = H/1 \cong H \longrightarrow A$ with the inclusion $i : H \longrightarrow A$, is the required right inverse for f.

Conversely, suppose $g : B \to A$ is a right inverse for f, which is a group morphism, and denote $H = g(B)$. We show that $\ker f$ is a (normal) complement of a subgroup H in A. Since now $\widetilde{g} : B \longrightarrow g(B)$ is an isomorphism, from $f \circ g = 1_B$ we derive $f \circ i \circ \widetilde{g} = 1_B$ and so $\widetilde{g} \circ f \circ i = 1_H$. According to Lemma 4 and its proof, the inclusion $i : H \longrightarrow A$ has a right inverse, and so $\ker(\widetilde{g} \circ f) = \ker f$ is a (normal) complement for H, as desired. ■

Remark 10 *Our general result in* ***Grp*** *implies the previous (known) result in* ***Ab****.*

Indeed, for abelian groups, $\ker f$ has a (normal) complement H in A means $A = H \oplus \ker f$, $H \cong B$ and one just has to take $C = \ker f$.

References

[1] Adamek J., Herrlich H., Strecker G. *Abstract and Concrete Categories, The Joy of Cats.* 2004 [http://katmat.math.uni-bremen.de/acc]