Mathematica Pannonica ***New Series*** **27 /NS 1/ (2021) 2, 81-88**
DOI: 10.1556/314.2021.00008

AKADÉMIAI KIADÓ

ON ZERO DETERMINANT MATRICES THAT ARE FULL

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Communicated by László Tóth

Original Research Paper

Received: Dec 25, 2020 • Accepted: Mar 22, 2021
First published online: October 8, 2021

ABSTRACT

Column-row products have zero determinant over any commutative ring. In this paper we discuss the converse. For domains, we show that this yields a characterization of pre-Schreier rings, and for rings with zero divisors we show that reduced pre-Schreier rings have this property.

Finally, for the rings of integers modulo n, we determine the 2×2 matrices which are (or not) full and their numbers.

KEYWORDS

full matrix, inner rank. zero determinant matrix, column-row decomposition, pre-Schreier ring

MATHEMATICS SUBJECT CLASSIFICATION (2020)

Primary 15B99; 16U30; Secondary 16Z99

1. INTRODUCTION

The *inner rank* of an $m \times n$ matrix over a ring is defined as the least integer r such that A can be expressed as a product of an $m \times r$ matrix and an $r \times n$ matrix. For example, over a division ring this notion coincides with the usual notion of rank. A square matrix is called *full* if its inner rank equals its order, and *non-full* otherwise.

It is easy to see that, over any commutative ring and for any positive integer n, the determinant of a n-column-n-row product is zero. Obviously, such products have inner rank 1.

In this paper, we discuss the converse, that is, we find conditions on a commutative ring, such that every zero determinant (square) matrix is a column-row product, that is, has inner rank 1.

It is also easy to see that this property fails for $n \geq 3$. Indeed, if one row has zero and nonzero entries, and in the column of one zero we have one nonzero entry, the matrix is not a column-row product. As an example, for any $n \geq 3$, the diagonal matrix $\mathrm{diag}(1, 1, 0, \ldots, 0)$ is a zero determinant matrix in $\mathbb{M}_n(R)$, which has no column-row decompositions, over any commutative, unital ring R.

This explains why, we mostly refer to 2×2 matrices. For these, full matrices have inner rank two (that is, these do not have column-row decompositions) and non-full (nonzero) matrices have inner rank 1 (that is, these have column-row decompositions).

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Over integral domains, this condition turns out to be a (new) characterization of pre-Schreier domains.

A *commutative* ring R is called *pre-Schreier*, if every nonzero element $r \in R$ is *primal*, i.e., if r divides xy, there are r_1, r_2 elements in R such that $r = r_1 r_2$, r_1 divides x and r_2 divides y.

Pre-Schreier domains were introduced by M. Zafrullah in [5]. A pre-Schreier integrally closed domain was called a *Schreier* domain by P. M. Cohn in [2]. Every GCD (greatest common divisor) domain is Schreier. In general, an irreducible element is primal if and only if it is a prime element. Consequently, in a pre-Schreier domain, every irreducible is prime.

The characterization is the following

THEOREM . Let R be a commutative unital ring. Consider the following conditions:

(i) every 2×2 zero determinant matrix over R is non-full;

(ii) R is pre-Schreier.

Then (i) implies (ii) and, if R is a domain, (ii) implies (i).

An example shows that the pre-Schreier condition is necessary in order to have such column-row decompositions.

For rings with zero divisors, we show that the property holds for reduced rings (i.e. without nonzero nilpotent elements), that is, we prove the following

THEOREM . All the zero determinant 2×2 matrices over any pre-Schreier reduced ring are non-full.

As already mentioned, over commutative rings *every non-full matrix has zero determinant.*

In the last section, we determine the full 2×2 matrices and their numbers over the rings $\mathbb{Z}_n$ (integers modulo n), for positive integers $n \geq 2$.

A problem *apparently* connected, is how to fit into our study the 2×2 matrices which have *dependent* rows (or columns).

Clearly, *every non-full matrix has dependent rows.* However the converse fails in general. As a special case of a result in the last section, $2I_2$ is full over $\mathbb{Z}_4$, but has dependent rows: $2\,\mathrm{row}_1(2I_2) = 2\,\mathrm{row}_2(2I_2) = 0$.

On the other hand, *rows may be dependent for a nonzero determinant matrix*: over $\mathbb{Z}_6$,

$$\det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right) = 3 \neq 0,$$

but $2\begin{bmatrix} 2 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

For a unital ring R, $U(R)$ denotes the set of all the units of R and $GL_n(R) = U(\mathbb{M}_n(R))$. For a finite set X, $|X|$ denotes the number of elements of X, and, for a positive integer k, $\left[\frac{k}{2}\right]$ denotes the integer part of $\frac{k}{2}$ (also called its "floor").

2. PROOF OF THE THEOREMS AND CONSEQUENCES

As already mentioned, we show that, the coincidence of zero determinant matrices and non-full matrices over (commutative) rings, characterizes pre-Schreier domains.

THEOREM 1. Let R be a commutative unital ring. Consider the following conditions:

(i) every zero determinant 2×2 matrix over R is non-full;

(ii) R is pre-Schreier.

Then (i) implies (ii) and, if R is a domain, (ii) implies (i).

Proof. Suppose $0 \neq r \in R$ divides xy. Then $xy = rt$ for some $t \in R$ and since $\begin{bmatrix} x & t \\ r & y \end{bmatrix}$ has zero determinant, it has a column-row decomposition, $\begin{bmatrix} s \\ u \end{bmatrix}\begin{bmatrix} a & b \end{bmatrix}$. Then $r = au$, $x = sa$ and $y = ub$, so a divides x and u divides y. Hence r is primal, as desired.

As for the converse, let R be a pre-Schreier domain and let $A = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ with $xt = yz$. First suppose, both x and z are nonzero. Since x divides yz there exist x_1, x_2, y_1, z_1 such that $x = x_1x_2$, $y = x_1y_1$ and $z = x_2z_1$. By cancellation, $xt = yz$ yields $t = y_1z_1$ and so $A = \begin{bmatrix} x_1 \\ z_1 \end{bmatrix} \begin{bmatrix} x_2 & y_1 \end{bmatrix}$.

Secondly, suppose $x \neq 0 = z$. Then, since $\det(A) = 0$, we get $t = 0$ and $A = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix}$.
In the remaining case, $A = \begin{bmatrix} 0 & y \\ 0 & t \end{bmatrix} = \begin{bmatrix} y \\ t \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$. □

REMARK. A similar weaker result, for integral domains and 2×2 matrices whose rank is 1 over the field of fractions, appears in [4].

EXAMPLE. It is well-known that $\mathbb{Z}[i\sqrt{5}] = \{m + ni\sqrt{5} : m, n \in \mathbb{Z}\}$ is (a domain which is) not UFD. Since 3 is irreducible but not a prime, $\mathbb{Z}[i\sqrt{5}]$ is also not (pre-)Schreier. It can be checked that over $\mathbb{Z}[i\sqrt{5}]$, the 2×2 zero determinant matrix $A = \begin{bmatrix} 3 & 1 - i\sqrt{5} \\ 1 + i\sqrt{5} & 2 \end{bmatrix}$ is full.

In rings with zero divisors we can prove the following

THEOREM 2. All the zero determinant matrices over any pre-Schreier reduced ring are non-full.

Proof. We use Andrunakievici-Ryabukhin theorem: a nonzero ring R is reduced if and only if R is a subdirect product of domains (see [1]). Recall that a ring R is said to be (represented as) a subdirect product of a family of rings $\{R_i : i \in I\}$ if there is a monomorphism (i.e., injective ring homomorphism) $f : R \longrightarrow \prod R_i$, and $f_i := \pi_i \circ f : R \longrightarrow R_i$ are surjective for each $i \in I$, where $\pi_i : \prod R_i \longrightarrow R_i$ is the canonical projection map.

Notice that for a monomorphism $f : R \longrightarrow \prod R_i$ of commutative rings as above, and a square matrix M over R, $f(\det(M)) = \det(f(M))$. Here if $M = [m_{jk}] \in \mathbb{M}_n(R)$ then $f(M)$ denotes $[f(m_{jk})] \in \mathbb{M}_n(\prod R_i) \cong \prod \mathbb{M}_n(R_i)$ and $\det(f(M)) = (\det(f_i(M))) \in \prod R_i$.

In particular $\det(M) = 0$ if and only if $\det((\pi_i \circ f)(M)) = 0$ for all $i \in I$.

Analogously, M is non-full over R if and only if $(\pi_i \circ f)(M)$ are non-full over R_i, for all $i \in I$. □

In particular,

COROLLARY 3. All the zero determinant matrices over any product of fields are non-full.

COROLLARY 4. Let n be a square-free positive integer. Then all the zero determinant matrices over $\mathbb{Z}_n$ are non-full.

3. NON-FULL MATRICES OVER INTEGERS MOD N

We first mention some useful auxiliary results.

Let $A, B \in \mathbb{M}_n(R)$. Then B is *equivalent* to A if there are units $P, Q \in GL_n(R)$ such that $B = PAQ$. Notice that, over any commutative ring R, if B is equivalent to A then rB is equivalent to rA, for any $r \in R$. It is easy to see that both the zero determinant and the non-full conditions are *invariant to equivalences.*

As a special case, if $u \in U(R)$, we can take P or Q as uI_2, so we can multiply matrices by units, without changing the properties just mentioned.

Denote by $\gcd(A)$ the greatest common divisor of the entries of the matrix A, if the GCD of the entries exists. When discussing these properties, if $\gcd(A) = u \in U(R)$, we can suppose $\gcd(A) = 1$, just multiplying the matrix by u^{-1}.

REMARK. If A is non-full, so is any multiple rA for any $r \in R$. *The converse fails*: it is easy to check that $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ does not have zero determinant over any (commutative) ring, unless $2 = 0$, but over $\mathbb{Z}_4$,
$2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \end{bmatrix}$.

PROPOSITION 5. Both the set of all zero determinant matrices and the set of all non-full matrices are closed under: multiplication, transpose, interchange of rows (or columns) and elementary row (or column) transformations, provided that multiplications of rows (or columns) by "scalars" are made only with units.

Proof. All properties are well-known for zero determinant matrices and are easy to check for non-full matrices. Right multiplication by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ interchanges the columns, left multiplication, interchanges rows. □

The following simple result will be useful in the sequel.

PROPOSITION 6. If a zero determinant matrix over any commutative ring R has a unit entry, it is non-full.

Proof. Interchanging rows and/or columns it suffices to check the NW corner unit entry case, that is, for any matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, suppose a is a unit. Then $A = \begin{bmatrix} 1 \\ a^{-1}c \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix}$, since $ad = bc$. □

EXAMPLE . Take $A = \begin{bmatrix} 6 & 3 \\ 10 & 0 \end{bmatrix}$ over $\mathbb{Z}_{30}$ (zero determinant with no unit entry). It is easy to check that A has no decomposition of types $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix}$, nor $\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix}$. However, A is non-full: $\begin{bmatrix} 6 & 3 \\ 10 & 0 \end{bmatrix} = \begin{bmatrix} 21 \\ 10 \end{bmatrix} \begin{bmatrix} 16 & 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \end{bmatrix} \begin{bmatrix} 4 & 27 \end{bmatrix}$. Notice that the first decomposition is of type $\begin{bmatrix} x \\ c \end{bmatrix} \begin{bmatrix} y & b \end{bmatrix}$. Such decompositions can be successfully used, when finding column-row decompositions, for zero determinant matrices over $\mathbb{Z}_n$, with square-free n.

Suppose $n = p_1^{r_1} \dots p_k^{r_k}$ with $r_1, \dots, r_l \geq 2$ and $r_{l+1}, \dots, r_k = 1$. Then $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{r_1}} \times \dots \times \mathbb{Z}_{p_l^{r_l}} \times \mathbb{Z}_{p_{l+1}} \times \dots \times \mathbb{Z}_{p_k}$ is a finite direct product of local rings (i.e. a semilocal ring), and fields. Moreover, all primes p_i are different. Using Theorem 2, in the finite direct product case, and Corollary 4, it follows that the determination of the zero determinant matrices which are (or not) full reduces to the case when n is the power (≥ 2) of a prime number.

Recall that if $n = p_1^{r_1} \dots p_k^{r_k}$, then a is nilpotent in $\mathbb{Z}_n$ if and only if a is divisible by $p_1 \dots p_k$. Since the rings $\mathbb{Z}_n$ are finite unital rings, a nonzero element is a unit if and only if it is not a zero divisor.

We mention that, since in the sequel we discuss $\mathbb{Z}_{p^k}$, which is not a domain, elements a, b are called *associates* if $b = au$ with some unit u (and not the usual a divides b and b divides a). This way, up to associates, the nonzero nilpotents of $\mathbb{Z}_{p^k}$ are p^m with $1 \leq m \leq k-1$.

LEMMA 7. Suppose $n = p^k$, $k > 1$. The matrices $\begin{bmatrix} p^m & 0 \\ 0 & p^l \end{bmatrix}$ with $1 \leq m, l \leq k-1$ are full over $\mathbb{Z}_n$. In particular, so are the matrices $p^m I_2$.

Proof. By contradiction, suppose such a matrix is non-full, that is $\begin{bmatrix} p^m & 0 \\ 0 & p^l \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix}$ with $1 \leq m, l \leq k-1$. Equivalently, the system

$$ax = p^m, ay = 0, bx = 0, by = p^l$$

is solvable over $\mathbb{Z}_{p^k}$. None of a, b, x, y is a unit (for instance, $a \in U(\mathbb{Z}_{p^k})$ implies $y = 0$ which contradicts $by = p^l$). Hence all a, b, x, y are associates of nilpotents (since $\mathbb{Z}_{p^k}$ is local) and since we can neglect units (by Proposition 5), denote $a = p^i$, $b = p^j$, $x = p^r$, $y = p^t$. Then $ax = p^i \cdot p^r$ implies $i + r = m$, $ay = p^i \cdot p^t$ imples $i + t \geq k$, $bx = p^j \cdot p^r$ implies $j + r \geq k$ and $by = p^j \cdot p^t$ implies $j + t = l$. From the first and last we have $i + j + r + t = l + m$. From the middle two we have $i + j + r + t \geq 2k$ and this yields $l + m \geq 2k$ (both $m, l \leq k - 1$). □

Since we merely discuss zero determinant matrices that are full, notice, by denial of Proposition 5, that if a zero determinant matrix is full, the matrices obtained from it, by interchanging rows (or columns), are also (zero determinant matrices that are) full. What changes is the sign of their determinants.

We also mention that if a matrix $A \in \mathbb{M}_2(\mathbb{Z}_{p^k})$ is a multiple of p, namely $A = pA_1$, then the entries of A_1 belong to $\mathbb{Z}_{p^{k-1}}$; if it is a multiple of p^2, say $A = p^2A_2$, the entries of A_2 belong to $\mathbb{Z}_{p^{k-2}}$, and so on. Finally, if $A = p^{k-1}A_{k-1}$, the entries of A_{k-1} belong to $\mathbb{Z}_p$. To simplify the wording, if $A = p^lA_l$ we will say that $A_l \in \mathbb{M}_2(\mathbb{Z}_{p^{k-l}})$ for any $1 \leq l \leq k-1$.

THEOREM 8. Let $A \in \mathbb{M}_2(\mathbb{Z}_{p^k})$ be a nonzero zero determinant matrix. Then A is full if and only if $\gcd(A)$ is a nonzero nilpotent of $\mathbb{Z}_{p^k}$ and, if $A = \gcd(A)A_1$, $\det(A_1) \neq 0$.

Proof. According to Proposition 6, since $\mathbb{Z}_{p^k}$ is local, only zero determinant matrices with (possibly zero) nilpotent entries may be full. Since not all entries are zero and the nonzero nilpotents are associated with p^m with $1 \leq m \leq k-1$, $\gcd(A)$ is a nonzero nilpotent of $\mathbb{Z}_{p^k}$.

Since A is nonzero, so is A_1. Moreover, from $\gcd(A_1) = 1$, it follows that A_1 must have at least *a unit entry*. Notice that if $\det(A_1) = 0$, by Proposition 6, A_1 is non-full and so is A. Hence $\det(A_1) \neq 0$.

Conversely, let $p^m = \gcd(A)$. Then $A = p^mA_1$ with $\gcd(A_1) = 1$, $\det(A_1) \neq 0$ and (as noted in the paragraph before the theorem) we can suppose $A_1 \in \mathbb{M}_2(\mathbb{Z}_{p^{k-m}})$. Again since $\mathbb{Z}_{p^k}$ is local, $\det(A_1)$ is a unit or else a nilpotent.

In the first case, $\det(A_1)$ is a unit, and showing that A is full, reduces to show that p^mI_2 is full, which follows from the previous lemma.

In the remaining case, $\det(A_1)$ is nonzero nilpotent.

As already noted, A_1 must have at least *a unit entry* which (we use Proposition 5), by interchanging rows and/or columns, we may suppose in the NW corner. Moreover, multiplying the first row by the inverse of the NW entry, we may suppose that the NW entry is 1. Further, we transform this matrix (as for the echelon form) into an equivalent diagonal matrix $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & \beta \end{bmatrix}$ with a nonzero nilpotent β. Clearly, $\det(A_1) = \det(A_2) = \beta \in N(\mathbb{Z}_{p^{k-m}})$.

Notice that, since $\det(A) = p^{2m}\det(A_1) = p^{2m}\beta = 0$, β is divisible by p^{k-2m}, $p^m\beta$ is divisible by p^{k-m} and so $p^m\beta$ is associated to a nonzero power of p, say p^l. But then A is equivalent to $p^mA_2 = \begin{bmatrix} p^m & 0 \\ 0 & p^l \end{bmatrix}$, which is full, again from the previous lemma. □

COROLLARY 9. Zero determinant matrices with only diagonal (or secondary diagonal) nonzero entries over $\mathbb{Z}_{p^k}$ are full.

REMARKS . 1. A careful observation of the proof of the previous theorem shows that the statement remains true over any *local commutative* ring such that *the ideal of nilpotent elements is principal.*

2. Since $\mathbb{Z}_n$ is reduced if and only if n is square-free, the previous theorem also shows that, for rings of integers modulo n, the converse of Corollary 4 holds. We were not able **to prove or disprove** the following statement: *a pre-Schreier ring, all whose* 2×2 *zero determinant matrices are non-full, is reduced.*

Notice that if $A = p^mA_1$ then $\det(A) = p^{2m}\det(A_1) = 0$ for any zero determinant matrix. Hence, $\det(A_1)$, and so A_1, can be units if and only if $2m \geq k$ (i.e. $m \geq \frac{k}{2}$).

Moreover, if $m = 1$, $\det(A_1)$ must be (associated to) p^{k-2}, if $m = 2$, $\det(A_1)$ must be (associated to) p^{k-3} or p^{k-4}, and so on.

Therefore we can list the full zero determinant matrices over $\mathbb{Z}_{p^k}$ as follows

THEOREM 10. Let p be a prime number. The only nonzero full zero determinant matrices A of $\mathbb{M}_2(\mathbb{Z}_{p^k})$, are the matrices of form p^mA_1 with $1 \leq m \leq k-1$ and $\gcd(A_1) = 1$, where:

(1) for $m = k-1$, A_1 is a unit of $\mathbb{M}_2(\mathbb{Z}_p)$;

(2) for $m = k-2$, A_1 belongs to $\mathbb{M}_2(\mathbb{Z}_{p^2})$ and has determinant associated to any nonzero nilpotent of $\mathbb{Z}_{p^2}$ or else, is a unit in $\mathbb{M}_2(\mathbb{Z}_{p^2})$;

(3) for $m = k-3$, A_1 belongs to $\mathbb{M}_2(\mathbb{Z}_{p^3})$ and has determinant associated to any nonzero nilpotent of $\mathbb{Z}_{p^3}$ or else, is a unit in $\mathbb{M}_2(\mathbb{Z}_{p^3})$;

⋮

($\left[\frac{k}{2}\right]$) for $m = k - \left[\frac{k}{2}\right]$, A_1 belongs to $\mathbb{M}_2\left(\mathbb{Z}_{p^{\left[\frac{k}{2}\right]}}\right)$ and has determinant associated to any nonzero nilpotent of $\mathbb{Z}_{p^{\left[\frac{k}{2}\right]}}$ or else, is a unit in $\mathbb{M}_2(\mathbb{Z}_{p^{\left[\frac{k}{2}\right]}})$;

($\left[\frac{k}{2}\right]+1$) for $m = k - \left[\frac{k}{2}\right] - 1$, A_1 belongs to $\mathbb{M}_2\left(\mathbb{Z}_{p^{\left[\frac{k}{2}\right]}}\right)$ and has determinant associated to any nonzero nilpotent of $\mathbb{Z}_{p^{\left[\frac{k}{2}\right]}}$ (no more units);

⋮

($k-2$) for $m = 2$, A_1 belongs to $\mathbb{M}_2(\mathbb{Z}_{p^{k-2}})$ and has (nilpotent) determinant associated to p^{k-3} or p^{k-4}.

($k-1$) for $m = 1$, A_1 belongs to $\mathbb{M}_2(\mathbb{Z}_{p^{k-1}})$ and has (nilpotent) determinant associated to p^{k-2}. □

As in [3], we shall use the following notations: $D_n(m,k) = \{A \in M_n(\mathbb{Z}_m) : \det A \equiv k \pmod{m}\}$ and $d_n(m,k) = |D_n(m,k)|$.

Recall from [3] that (for $n = 2$) the number of 2×2 matrices over $\mathbb{Z}_{p^r}$ of determinant p^l ($r > l$) is $d_2(p^r; p^l) = p^{3r} + p^{3r-1} - p^{3r-l-1} - p^{3r-l-2}$, with $\left|GL_2(\mathbb{Z}_p)\right| = d_2(p; p^0) = (p^2-1)(p^2-p)$. In particular,

$$d_2(p^r; p^{r-1}) = p^{3r} + p^{3r-1} - p^{2r} - p^{2r-1}.$$

Moreover, also from [3], $d_2(p^k; 0) = p^{2k-1}(p^{k+1} + p^k - 1)$ is the number of zero determinant matrices over $\mathbb{Z}_{p^k}$.

COROLLARY 11. The number of full zero determinant matrices over $\mathbb{Z}_{p^k}$ is

$$\left|GL_2(\mathbb{Z}_p)\right| + (p-1)[d_2(p^2;p) + d_2(p^3;p^2) + \ldots + d_2(p^{k-1};p^{k-2})].$$

Proof. We just count the matrices listed in the previous theorem. Here $p-1$ is the number of associates for any nilpotent.

(1) $\left|GL_2\left(\mathbb{Z}_p\right)\right|$;

(2) $(p-1)d_2\left(p^2;p\right) + \left|GL_2\left(\mathbb{Z}_{p^2}\right)\right|$

(3) $(p-1)d_2\left(p^3;p^2\right) - \left|GL_2\left(\mathbb{Z}_{p^2}\right)\right| + \left|GL_2\left(\mathbb{Z}_{p^3}\right)\right|$

(4) $(p-1)d_2\left(p^4;p^3\right) - \left|GL_2\left(\mathbb{Z}_{p^3}\right)\right| + \left|GL_2\left(\mathbb{Z}_{p^4}\right)\right|$

⋮

($\left[\frac{k}{2}\right]$) $(p-1)d_2\left(p^{\left[\frac{k}{2}\right]}; p^{\left[\frac{k}{2}\right]-1}\right) - \left|GL_2\left(\mathbb{Z}_{p^{\left[\frac{k}{2}\right]-1}}\right)\right| + \left|GL_2\left(\mathbb{Z}_{p^{\left[\frac{k}{2}\right]}}\right)\right|$

($\left[\frac{k}{2}\right]+1$) $(p-1)d_2\left(p^{\left[\frac{k}{2}\right]+1}; p^{\left[\frac{k}{2}\right]}\right) - \left|GL_2\left(\mathbb{Z}_{p^{\left[\frac{k}{2}\right]}}\right)\right|$

($\left[\frac{k}{2}\right]+2$) $(p-1)d_2\left(p^{\left[\frac{k}{2}\right]+2}; p^{\left[\frac{k}{2}\right]+1}\right)$

⋮

($k-1$) $(p-1)d_2(p^{k-1}; p^{k-2})$. □

COROLLARY 12. The number of full zero determinant matrices over $\mathbb{Z}_{p^k}$ is

$$\frac{1}{p^3-1}\left(p^{3k+1}-p^{3k-1}-p^{2k+2}+p^{2k+1}+p^2-p\right).$$

Proof. Repeatedly using $d_2(p^r;p^{r-1}) = p^{3r}+p^{3r-1}-p^{2r}-p^{2r-1}$, the sum S in the previous corollary is $(p^2-1)(p^2-p)+(p-1)\sigma$ where $\sigma = \sum_{i=2}^{k-1}(p^{3i}+p^{3i-1}-p^{2i}-p^{2i-1})$. Separatedly adding these four geometric progressions gives

$$\sigma = \frac{p^{3k}+p^{3k-1}-p^6-p^5}{p^3-1} - \frac{p^{2k}+p^{2k-1}-p^4-p^3}{p^2-1}.$$

Then

$$S = \frac{1}{p^3-1}\left[(p^2-1)(p^2-p)(p^3-1)+(p^{3k}+p^{3k-1}-p^6-p^5)(p-1)-(p^{2k-1}-p^3)(p^3-1)\right]$$

which finally yields

$$S = \frac{1}{p^3-1}\left(p^{3k+1}-p^{3k-1}-p^{2k+2}+p^{2k+1}+p^2-p\right).$$ □

REMARK. In order to count the number of non-full matrices over $\mathbb{Z}_{p^k}$, it suffices to subtract from the total number of zero determinant matrices, that is $d_2(p^k;0) = p^{2k-1}(p^{k+1}+p^k-1)$, the number in the previous corollary and add 1 (the zero matrix). This number is

$$\frac{1}{p^3-1}(p+1)(p^2-1)(p^{3k}-1)+1.$$

Denote by $\mathbb{O}_2(\mathbb{Z}_n)$ the set of all the zero determinant matrices over $\mathbb{Z}_n$, by $\mathbb{D}_2(\mathbb{Z}_n)$ the set of all the non-full matrices over $\mathbb{Z}_n$ and by $\mathbb{ND}_2(\mathbb{Z}_n) = \mathbb{O}_2(\mathbb{Z}_n)\setminus\mathbb{D}_2(\mathbb{Z}_n)$.

Moreover, we introduce the following numerical functions: $f,g:\mathbb{Z}_+^*\longrightarrow\mathbb{Z}_+^*$ defined by $f(n) = |\mathbb{O}_2(\mathbb{Z}_n)|$ and $g(n) = |\mathbb{D}_2(\mathbb{Z}_n)|$. Observe that, according to Corollary 4, for any square-free positive integer n, $f(n) = g(n)$ and for any positive integer n, $f(n)\ge g(n)$. Obviously, $|\mathbb{ND}_2(\mathbb{Z}_n)| = f(n)-g(n)$.

First an easy result

LEMMA 13. (i) If n, m are coprime positive integers, then $|\mathbb{O}_2(\mathbb{Z}_{nm})| = |\mathbb{O}_2(\mathbb{Z}_n)|\cdot|\mathbb{O}_2(\mathbb{Z}_m)|$, i.e. $f(nm) = f(n)f(m)$ and $|\mathbb{D}_2(\mathbb{Z}_{nm})| = |\mathbb{D}_2(\mathbb{Z}_n)|\cdot|\mathbb{D}_2(\mathbb{Z}_m)|$, i.e. $g(nm) = g(n)g(m)$.

(ii) Let $n = p_1^{r_1}\dots p_l^{r_l}p_{l+1}\dots p_m$ with $r_1,\dots,r_l\ge 2$ and different primes p_i $(1\le i\le m)$. Then $g(n) = g(p_1^{r_1})\dots g(p_l^{r_l})f(p_{l+1})\dots f(p_m)$ gives the number of non-full matrices over $\mathbb{Z}_n$.

Proof. (i) If n, m are coprime, $\mathbb{Z}_{nm}\cong\mathbb{Z}_n\times\mathbb{Z}_m$, the direct product. Moreover, $\mathbb{M}_2(\mathbb{Z}_{nm})\cong\mathbb{M}_2(\mathbb{Z}_n)\times\mathbb{M}_2(\mathbb{Z}_m)$ and it is readily seen that this isomorphism preserves zero determinant matrices and non-full matrices, respectively.

(ii) Just use the fact that f, g are multiplicative (i.e. (i)). Then

$$|\mathbb{D}_2(\mathbb{Z}_n)| = g(n) = g(p_1^{r_1})\dots g(p_l^{r_l})g(p_{l+1})\dots g(p_m) = g(p_1^{r_1})\dots g(p_l^{r_l})f(p_{l+1})\dots f(p_m).$$ □

Since the zero determinant matrices coincide with the non-full matrices over $\mathbb{Z}_q$ for any square-free q, it was easy to foresee the following

PROPOSITION 14. Let $n = p^kq$ with square-free q. Then

$$\frac{\left|\mathbb{O}_2(\mathbb{Z}_{p^kq})\right|}{\left|\mathbb{ND}_2(\mathbb{Z}_{p^kq})\right|} = \frac{\left|\mathbb{O}_2(\mathbb{Z}_{p^k})\right|}{\left|\mathbb{ND}_2(\mathbb{Z}_{p^k})\right|},$$

that is,

$$\frac{f(p^kq)}{f(p^kq)-g(p^kq)} = \frac{f(p^k)}{f(p^k)-g(p^k)}.$$

Proof. By Lemma 13, $\left|\mathbb{O}_2(\mathbb{Z}_{p^k q})\right| = \left|\mathbb{O}_2(\mathbb{Z}_{p^k})\right| \cdot \left|\mathbb{O}_2(\mathbb{Z}_q)\right|$, that is $f(p^k q) = f(p^k)f(q)$, so the equality in the statement reduces to $f(p^k q) - g(p^k q) = f(q)(f(p^k) - g(p^k))$. Equivalently, $g(p^k q) = g(p^k)f(q) = g(p^k)g(q)$, which holds by the previous lemma. □

REMARK . Observe that, by [3], $\left|\mathbb{O}_2(\mathbb{Z}_{p^k})\right| = p^{2k-1}(p^{k+1} + p^k - 1)$ and $\left|\mathbb{ND}_2(\mathbb{Z}_{p^k})\right|$ is the number determined in Corollary 12.

ACKNOWLEDGEMENTS

Thanks are due to Simion Breaz for fruitful discussions on this subject and for pointing out the pre-Schreier condition and to the referee for careful reading and comments which improved our presentation.

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