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Fine rings: A new class of simple rings

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A nonzero ring is said to be *fine* if every nonzero element in it is a sum of a unit and a nilpotent element. We show that fine rings form a proper class of simple rings, and they include properly the class of all simple artinian rings. One of the main results in this paper is that matrix rings over fine rings are always fine rings. This implies, in particular, that any nonzero (square) matrix over a division ring is the sum of an invertible matrix and a nilpotent matrix.

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1. Introduction

The work in this paper is prompted by the idea of looking at the three sets $\mathrm{U}(R)$, idem (R), and nil (R) in any (unital) ring R, which denote, respectively, the unit group, the set of idempotents, and the set of nilpotent elements in R. In the last four decades, an additive theory has emerged in the study of these three interesting sets. In [22], Nicholson defined a ring element $a \in R$ to be *clean* if it can be written in the form $e + u$ where $e \in \mathrm{idem}(R)$ and $u \in \mathrm{U}(R)$. If every $a \in R$ is clean, R is said to be a *clean ring.* Prompted by this, Diesl [11] defined a ring element $b \in R$ to be *nil-clean* if $b = e + t$ for some $e \in \mathrm{idem}(R)$ and $t \in \mathrm{nil}(R)$. If every $b \in R$

is nil-clean, R is said to be a *nil-clean ring*. It is easy to see that nil-clean rings are always clean [11, Proposition 3.4], though in general, clean rings need not be nil-clean.

Guided by the definitions in the last paragraph, we investigate in this work the third (and last) possible way of adding a pair of elements from two of the three sets $\mathrm{U}(R)$, idem (R), and $\mathrm{nil}(R)$ above. Thus, we define a nonzero ring element $a \in R$ to be *fine* if $a = u + t$ for some $u \in \mathrm{U}(R)$ and some $t \in \mathrm{nil}(R)$. (The intuitive idea behind $a \in R\backslash(0)$ being fine is that a is "invertible modulo a nilpotent element.") Any equation of the form $a = u+t$ will be called a *fine decomposition* of $a \in R\backslash(0)$, and we will write $\Phi(R)$ for the set of fine elements in $R\backslash(0)$. (The reason for stipulating that $0 \notin \Phi(R)$ is that if there exists an equation $0 = u + t$ where $u \in \mathrm{U}(R)$ and $t \in \mathrm{nil}(R)$, then $u = -t$ would imply that $R = 0$, which is a trivial case.) A nonzero ring R with $R\backslash(0) = \Phi(R)$ will be called a *fine ring*. For instance, any division ring is a fine ring, and the converse statement holds for reduced rings.

It turns out that fine rings are quite different from the clean rings and the nil-clean rings. One very special feature of fine rings is that *they are always simple rings*; see Theorem 2.3(2). However, not all simple rings are fine, so fine rings form a new proper subclass of simple rings.

It is classically well known that matrix rings over simple rings are simple; see, e.g. [19, Theorem 3.1]. The "analogy" between fine rings and simple rings led us to the discovery of the following main result in this paper.

Main Theorem. *If S is a fine ring, so is $\mathbb{M}_n(S)$ for any integer $n \geq 1$.*

To put things in perspective, we should point out that the result above was known to be true for clean rings by the work of Han and Nicholson in [15], but not known to be true or false for nil-clean rings. (See, however, the paper [5] of Breaz, Călugăreanu, Danchev and Micu.)

Since division rings are fine, the Main Theorem implies that *all simple artinian rings are fine rings*, although fine rings in general need not be (left or right) artinian. Thus, the study of fine rings amounts to working with a new class of simple rings which generalizes the well known classical family of the artinian simple rings.

From the viewpoint of linear algebra and operator theory, the fact that $\mathbb{M}_n(D)$ is a fine ring for every division ring D amounts to the fact that any nonzero linear operator φ on the vector space D^n can be suitably "perturbed" by a nilpotent operator to become invertible. Alternatively, φ can also be suitably "perturbed" by an invertible operator to become nilpotent. We have not been able to find a reference for either one of these facts in the linear algebra literature. On the other hand, it would be of interest to investigate in what way, and to what extent, these facts can be extended to nonzero operators on infinite-dimensional linear spaces (such as Hilbert spaces).

In Sec. 2 of this paper, we collect a few basic properties of fine elements and fine rings, beginning with a quick proof of the simplicity of fine rings. We also show that all fine rings R have the property that $R = \mathrm{U}(R) + \mathrm{U}(R)$, except in the case where $R \cong \mathbb{F}_2$. After this preparatory work, the Main Theorem stated above is proved in Sec. 3. In Sec. 4, we prove some partial results toward the Morita invariance of fine rings, showing for instance that a matrix ring over a commutative ring S is fine if and only if S is a field. The last section, Sec. 5, collects a number of results on fine and clean matrices in $\mathbb{M}_2(S)$ over a commutative ring S. For instance, it is shown that, over a commutative domain S, any fine matrix of the form $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ is clean, although "fine" and "clean" are shown to be (in general) independent notions in $\mathbb{M}_2(S)$. To stimulate future work, five open questions are raised in (2.10), (3.14), (3.15), (4.1) and (5.19).

The terminology and notations introduced so far in this Introduction will be used freely throughout the paper. Other standard terminology and conventions in ring theory follow mainly those in [18–20]. Whenever it is more convenient, we will use the widely accepted shorthand "iff" for "if and only if" in the text.

2. Basic Facts on Fine Elements and Fine Rings

Before we proceed to the study of fine elements and fine rings, we would like to point out something about them that is related to the issue of commutativity. Following Nicholson's definition of strongly clean elements in [23] and Diesl's definition of strongly nil-clean elements in [11], it is natural to define a nonzero element $a \in R$ to be *strongly fine* if there exists a fine decomposition $a = u + t$ such that $ut = tu$. If $R \neq 0$ and all elements in $R\backslash(0)$ are strongly fine, we say that R is a *strongly fine ring.* Similarly prompted by the work of Nicholson and Zhou in [24], we define a nonzero ring R to be *uniquely fine* if every element in $R\backslash(0)$ has a unique fine decomposition. While these are all natural definitions, they turn out not to amount to anything really new, as the following easy result shows.

Proposition 2.1. *For any ring R, the following statements hold.*

(1) *An element $a \in R\backslash(0)$ is strongly fine iff $a \in \mathrm{U}(R)$.*
(2) *R is strongly fine iff R is a division ring.*
(3) *R is fine and unit-central* (*i.e. all units are central*) *iff it is a field.*
(4) *R is uniquely fine iff R is a division ring.*

Proof. (1) Of course, units in a nonzero ring are always strongly fine. Conversely, if $a \in R\backslash(0)$ has a strongly fine decomposition $u + t$, then $a = u\,(1 + u^{-1}t) \in \mathrm{U}(R)$ (since $ut = tu$ implies that $u^{-1}t \in \mathrm{nil}(R)$). This proves (1), which clearly implies (2).

(3) It suffices to prove the "only if" part, so assume that R is fine and unit-central. Clearly, R is then strongly fine, so it is a division ring by (2). Since all units are central, R is a field.

(4) Again, it suffices to prove the "only if" part, so assume R is uniquely fine. If we can show that R is a *reduced ring,* it will be a division ring (as we have pointed out in Sec. 1). For any $t \in \mathrm{nil}(R)$, we have $1 + t \neq 0$ (since $R \neq 0$). From the two fine decompositions $1 + t = (1 + t) + 0$, we conclude that $t = 0$. $\square$

Next, we will make a series of observations about fine elements and fine rings. In the following, $Z(R)$ denotes the center of a ring R, and $\mathrm{rad}(R)$ denotes the Jacobson radical of R.

Proposition 2.2. (1) *For any nonzero ring R, $\Phi(R) \cap Z(R) = \mathrm{U}(Z(R))$.*
(2) $v \in \mathrm{U}(R) \Rightarrow v\Phi(R)\, v^{-1} = \Phi(R)$. *If $v \in \mathrm{U}(R) \cap Z(R)$, then $v\Phi(R) = \Phi(R)$.*
(3) *If $R \neq 0$ and $\mathrm{nil}(R) \subseteq \mathrm{rad}(R)$, then $\Phi(R) = \mathrm{U}(R)$. (In particular, this equality holds for all local rings and all nonzero 2-primal rings,[a] including all nonzero commutative rings.)*

Proof. (1) The inclusion "$\supseteq$" is trivial. Conversely, for any $a \in \Phi(R) \cap Z(R)$, take a fine decomposition $a = u + t$. Noting that u commutes with $t = a - u$, we have $a = u\,(1 + u^{-1}t) \in \mathrm{U}(R) \cdot \mathrm{U}(R) \subseteq \mathrm{U}(R)$. Thus, $a \in \mathrm{U}(Z(R))$.

(2) The first conclusion is clear. The second conclusion follows from the fact that if $a = u + t$ is a fine decomposition of $a \in \Phi(R)$, then $va = vu + vt$ is a fine decomposition of va for any $v \in \mathrm{U}(R) \cap Z(R)$.

(3) Finally, under the assumptions in (3), if $a \in R$ has a fine decomposition $u + t$, then $t \in \mathrm{nil}(R) \Rightarrow t \in \mathrm{rad}(R)$. Therefore, $a \in u + \mathrm{rad}(R) \subseteq \mathrm{U}(R)$. □

Theorem 2.3. (1) *If $a \in \Phi(R)$, then $RaR = R$.*
(2) *Any fine ring R is a simple ring.*
(3) *For any unital ring homomorphism $f : R \to S$ where $S \neq 0$, $f(\Phi(R)) \subseteq \Phi(S)$.*
(4) *Let I be a nil ideal of R, and let $a \in R$. Then $a \in \Phi(R)$ iff $\overline{a} \in \Phi(R/I)$.*

Proof. (1) Let $a = u + t$ be a fine decomposition. In the factor ring $\overline{R} = R/RaR$, the unit $\overline{u} = -\overline{t}$ is nilpotent. This implies that $\overline{R} = \{\overline{0}\}$; that is, $RaR = R$. (Alternatively, $(a - u)^n = t^n = 0$ for some integer $n \geq 1$. Expanding the nth power, we have $u^n \in RaR$, so $RaR = R$.)

(2) If R is a fine ring, (1) implies that $RaR = R$ for every nonzero $a \in R$. This means precisely that R is a simple ring.

(3) Let $a \in \Phi(R)$. Then $RaR = R$ by (1). Since $f(R) \neq 0$, we have $f(a) \neq 0$. After knowing this, it is clear that $f(a) \in \Phi(S)$.

(4) The "only if" part follows easily from (3). Conversely, if $\overline{a} \in \Phi(R/I)$, take a fine decomposition $\overline{a} = \overline{u} + \overline{t}$ in R/I. Since $I \subseteq \mathrm{rad}(R)$, we have $u \in \mathrm{U}(R)$. Moreover, $\overline{a - u} = \overline{t} \in \mathrm{nil}(R/I)$ implies that $a - u \in \mathrm{nil}(R)$ (since I is a nil ideal). Thus, $a \in u + \mathrm{nil}(R)$. Clearly, $a \neq 0$, so we have $a \in \Phi(R)$. □

Corollary 2.4. *A local ring $(R, \mathfrak{m})$ is a fine ring iff R is a division ring.*

Proof. The "if" part was already pointed out in Sec. 1, and the "only if" part is clear from both Theorem 2.3(2) and Proposition 2.2(3). (For a somewhat more general conclusion, see Corollary 3.11.) □

[a] A ring R is called 2-*primal* if $\mathrm{nil}(R)$ is contained in the prime radical of R; see [19, p. 195]

In contrast to Theorem 2.3(2), it is easy to see that simple rings need not be fine. Indeed, if we take any simple domain R that is not a division ring (e.g. a Weyl domain in characteristic zero as in [19, (3.17)], or a twisted Laurent polynomial ring $k\,[x, x^{-1}; \sigma]$ as in [19, (3.19)] where σ is an automorphism of infinite order on a field k), then R cannot be fine since $\mathrm{nil}(R) = 0$. However, this is not the only way in which one can construct non-fine simple rings. A rather different construction can be based upon the observation below.

Proposition 2.5. *If* $\mathrm{nil}(R)$ *is a nonzero additive subgroup in a ring* R, *then* R *is not a fine ring.*

Proof. Fix an element $t_0 \in \mathrm{nil}(R)\backslash(0)$. If R is a fine ring, then $t_0 = u + t$ for some $u \in \mathrm{U}(R)$ and $t \in \mathrm{nil}(R)$. The hypothesis implies that $u = t_0 - t \in \mathrm{nil}(R)$, so we must have $u = 0$. Thus $R = (0)$, which contradicts the fact that $t_0 \neq 0$. $\square$

Example 2.6. Using the work of Dubrovin [12], Chebotar, Lee and Puczylowski have constructed in [9, Theorem 10] a (unital) simple ring R in which $\mathrm{nil}(R)$ is a *nonzero subring.* By Proposition 2.5, R cannot be a fine ring.

Example 2.7. As we have mentioned in Sec. 1 (and will prove in Corollary 3.11), any artinian simple ring is fine. However, there do exist fine rings that are (simple but) *not* left or right artinian. To name such an example, we use the easily verified fact that if a ring R is a direct limit of a directed system of fine rings $\{R_i\}$, then R is itself a fine ring. In [19, pp. 39–40], there is an example of an ascending chain of simple artinian rings $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$ whose union R is a *non-artinian* simple ring. Since each R_i is a fine ring, so is R.

Next, we recall (e.g. from [14]) that, for a positive integer n, a ring R is said to have the n-*sum property* if every element of R is a sum of at most n units. (See also [4], where a slightly different terminology was used.) Part (1) of the following result implies that any fine ring has the 2-sum property. More precisely, part (2) shows that, with only one exception (namely, the field $\mathbb{F}_2$), fine rings are 2-*good* in the sense of Vámos [25] (namely, every element is a sum of two units).

Theorem 2.8. *For any fine ring* R, *the following statements hold*:

(1) $R = \{1\} \cup (\mathrm{U}(R) + \mathrm{U}(R))$.
(2) R *is* 2-*good iff* $|R| \neq 2$.

Proof. (1) Consider any $a \in R$. If $a \neq 1$, then $a - 1 \in \Phi(R)$, so it has a fine decomposition $u + t$. Thus, $a = u + (1 + t) \in \mathrm{U}(R) + \mathrm{U}(R)$.

(2) It suffices to prove the "if" part, so assume that $|R| > 2$. In view of (1), we need to only show that $1 \in \mathrm{U}(R) + \mathrm{U}(R)$. *First assume that* $\mathrm{U}(R) = \{1\}$. For any $a \in R\backslash\{0\}$, we have a fine decomposition $a = u + t$, so $a = 1 + t \in \mathrm{U}(R)$, and hence $a = 1$. This gives $|R| = 2$, which is not the case. Thus, there exists a unit $v \neq 1$.

Taking a fine decomposition $1 - v = u' + t'$, we have $w := u' + v = 1 - t' \in \mathrm{U}(R)$, and so $1 = w^{-1}u' + w^{-1}v \in \mathrm{U}(R) + \mathrm{U}(R)$, as desired. $\square$

Remark 2.9. As an interesting example for Theorem 2.8(2), let $R = \mathbb{M}_n(D)$ where D is a division ring. As we have mentioned earlier, R is a fine ring. The fact that R is 2-good except in the case where $n = 1$ and $D \cong \mathbb{F}_2$ (simultaneously) was first proved by Zelinsky in [26]. In view of this, we may think of Theorem 2.8(2) as a "fine analogue" of Zelinsky's classical result on simple artinian rings.

Theorems 2.3(2), 2.8(2) and Remark 2.9 led us to the following.

Question 2.10. *If a simple ring R is* 2*-good, is R necessarily a fine ring?*

The answer is very possibly "no", but we are not aware of any counterexamples. A weaker and somewhat more specific form of this question will be posed in (3.15)(A).

In [6, 10], a ring R is called a *UU ring* if all units in R are unipotent; i.e. if $\mathrm{U}(R) \subseteq 1 + \mathrm{nil}(R)$. In trying to determine which fine rings are UU, we find an easy answer in the following result.

Proposition 2.11. *A ring R is fine and UU iff $R \cong \mathbb{F}_2$.*

Proof. The "if" part is trivial. Conversely, let R be UU and fine, but assume, for the moment, that $|R| > 2$. In the proof of Theorem 2.8(2), we have shown that $1 = u_1 + u_2$ for some $u_1, u_2 \in \mathrm{U}(R)$. Since R is UU, $u_1 - 1 = -u_2 \in \mathrm{nil}(R) \cap \mathrm{U}(R)$. This implies that $R = 0$, which is not the case. Therefore, we must have $R \cong \mathbb{F}_2$. $\square$

We close this section by noting the following curious result on sums of *two* fine elements in matrix rings.

Proposition 2.12. *If $R = \mathbb{M}_n(S)$ where $n \geq 2$ and S is any nonzero ring, any matrix M is a sum of two fine matrices.*

Proof. We can write $M = D + T_1 + T_2$ where D is a diagonal matrix, T_1 is strictly upper triangular, and T_2 is strictly lower triangular. By [16], we can further write $D = U_1 + U_2$ where each U_i is invertible. Clearly, $U_1 + T_1 \neq 0 \neq U_2 + T_2$, so $M = (U_1 + T_1) + (U_2 + T_2) \in \Phi(R) + \Phi(R)$. $\square$

3. Matrix Rings Over Fine Rings

Since matrix rings over simple rings are simple (see, e.g. [19, Theorem 3.1]), it is natural to ask whether this statement has a valid analogue *for fine rings*. The goal of this section is to prove the following Main Theorem of this paper.

Theorem 3.1. *For a fine ring S, any matrix ring $\mathbb{M}_n(S)$ is fine.*

Throughout this section (except in Corollary 3.11 and Remark 3.12), we let $R := \mathbb{M}_n(S)$ where S is a nonzero ring (not assumed to be fine unless it is explicitly so stated). To investigate the fineness of the matrices in R, the following basic observation will be crucial.

Proposition 3.2. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in R = \mathbb{M}_n(S)$, *where* $A \in \mathbb{M}_k(S)$ *and* $D \in \mathbb{M}_{n-k}(S)$. *If* $A \in \Phi(\mathbb{M}_k(S))$ *and* $D \in \Phi(\mathbb{M}_{n-k}(S))$, *then* $M \in \Phi(R)$.

Proof. Let $A = U + T$ and $D = U' + T'$ be fine decompositions in $\mathbb{M}_k(S)$ and $\mathbb{M}_{n-k}(S)$ respectively. Then

$$M = \begin{pmatrix} U & 0 \\ C & U' \end{pmatrix} + \begin{pmatrix} T & B \\ 0 & T' \end{pmatrix}. \tag{$*$}$$

Here, the first matrix on the right-hand side is invertible in R. On the other hand, a high power of the second matrix on the right-hand side is strictly upper triangular, so the second matrix is nilpotent in R. Thus, $(*)$ above is a fine decomposition for M in R. □

Remark 3.3. Perhaps not surprisingly, a more abstract formulation of Proposition 3.2 is possible in terms of general Peirce decompositions *in an arbitrary ring* R. To be more precise, if e, f are complementary idempotents in R, and $a \in R$ is such that $eae \in \Phi(eRe)$ and $faf \in \Phi(fRf)$, we have shown in [7] that $a \in \Phi(R)$.

Applying Proposition 3.2 with an induction on the number of diagonal blocks of a block matrix, we arrive at the following result.

Theorem 3.4 (Block Theorem). *If* $M = (A_{ij}) \in R = \mathbb{M}_n(S)$ *is a block matrix where the blocks* $\{A_{ij}\}$ *are such that each* A_{ii} *is a fine square matrix, then* $M \in \Phi(R)$.

The discussions above suggest strongly that Theorem 3.1 should also be provable by induction on n. For such an induction to work, we will need to be able to "create" nonzero diagonal blocks out of a given nonzero matrix. To facilitate such a creation process, it will be convenient for us to use the following somewhat informal terminology.

Definition 3.5. For $n \geq 2$, we say that a matrix $\begin{pmatrix} A & \beta \\ \gamma & d \end{pmatrix} \in R$ is "in good form" if $A \in \mathbb{M}_{n-1}(S)$ is nonzero and $d \in S$ is also nonzero.

In view of Proposition 2.2(2), we can often work with elements of $R = \mathbb{M}_n(S)$ "up to conjugation". Throughout this paper, when we say "conjugating M by an invertible matrix V", we will mean "passing from M to VMV^{-1}". In the next few results, we shall investigate the possibility of bringing a nonzero matrix into good form by similarity transformations. We start with the case of 2×2 matrices over S.

Lemma 3.6. *Suppose there exists an equation* $1 = u + v \in S$ *where* $u, v \in \mathrm{U}(S)$. *Then any matrix* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *with* $a \neq 0$ *is similar to a matrix in good form.*

Proof. We may assume that $d = 0$ (for otherwise M is already in good form). To get the desired conclusion, we will consider the following three cases.

Case 1. $b = c = 0$. Here, conjugating M by $\begin{pmatrix} u & 1 \\ -v & 1 \end{pmatrix} \in \mathrm{GL}_2(S)$ gives $\begin{pmatrix} ua & -ua \\ -va & va \end{pmatrix}$, which is in good form (since $ua \neq 0 \neq va$).

Case 2. $c \neq 0$. Here, conjugating M by $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(S)$ gives $\begin{pmatrix} a-c & a+b-c \\ c & c \end{pmatrix}$. If $c \neq a$, this matrix is in good form, and we are done. If $c = a$, conjugating M by $\begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix}$ gives $\begin{pmatrix} au^{-1} & * \\ * & -au^{-1}v \end{pmatrix}$, which is in good form so we are also done. In particular, for $b = 0$, this argument would take care of the case $\begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix}$. Thus, it would have also taken care of the case $\begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix}$.

Case 3. $c = 0$. Here, in view of Case 1, we may assume that $b \neq 0$. Conjugating M by $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(S)$ gives $\begin{pmatrix} a-b & b \\ a-b & b \end{pmatrix}$. If $a \neq b$, this matrix is in good form. The remaining case $M = \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix}$ was covered by the last remark in Case 2 above. $\square$

Corollary 3.7. *Suppose there exists an equation $u + v = 1 \in S$ where $u, v \in \mathrm{U}(S)$. Then any nonzero matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(S)$ is similar to a matrix in good form.*

Proof. If $a \neq 0$, Lemma 3.6 applies. If $d \neq 0$ instead, we can conjugate M into $\begin{pmatrix} d & c \\ b & a \end{pmatrix}$ to go back to the first case. Now assume $a = d = 0$, and say $c \neq 0$. Conjugating $M = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ takes M to $\begin{pmatrix} c & b-c \\ c & -c \end{pmatrix}$, which is in good form. $\square$

Remark 3.8. Note that the assumption $1 \in \mathrm{U}(S) + \mathrm{U}(S)$ cannot be dropped from the above corollary. Indeed, if $S = \mathbb{F}_2$, the four matrices $M_i = \begin{pmatrix} 1 & * \\ * & 0 \end{pmatrix}$ cannot be similar to a matrix in good form since the latter matrix has trace 0, while each M_i has trace 1.

Next, we proceed to the case of $n \times n$ matrices.

Proposition 3.9. *Suppose there exists an equation $u + v = 1 \in S$ where $u, v \in \mathrm{U}(S)$. Then for any $n \geq 2$, any nonzero matrix $M = \begin{pmatrix} A & \beta \\ \gamma & d \end{pmatrix} \in R = \mathbb{M}_n(S)$ (where $d \in S$ and $A \in \mathbb{M}_{n-1}(S)$) is similar to a matrix in good form.*

Proof. We induct on n, the case $n = 2$ being already covered by Corollary 3.7. Working with $n \geq 3$, we may assume that the conclusion holds for nonzero matrices in $\mathbb{M}_{n-1}(S)$. We consider the following two cases.

Case 1. $A \neq 0$. Here, we may assume (after a conjugation) that A is in good form. Then the northeast $(n-2) \times (n-2)$ block of M is nonzero, and the southeast 2×2 block A' of M has $(1,1)$-entry nonzero. By Lemma 3.6, we can conjugate A' into a matrix in good form. Thus, we can conjugate M itself into a matrix in good form.

Case 2. We may now assume that $A = 0$. By the same token, we may assume that the southeast $(n-1) \times (n-1)$ block of M is also zero. Thus, all entries of M are zero except possibly its $(n,1)$-entry c and its $(1,n)$-entry b. Without loss of generality, say $c \neq 0$. Here, we can apply again the trick used in the proof of Corollary 3.7; that is, conjugating M by the elementary matrix $I_n + E_{1n}$ takes M into a matrix with northeast $(n-1) \times (n-1)$ block *nonzero,* so we are back to Case 1. $\square$

We are now ready to present the following.

Proof of Theorem 3.1. We let $R = \mathbb{M}_n(S)$ as usual, and assume that S is a fine ring. The proof will be by induction on n, the case $n = 1$ being trivial.

We will first work in the case where $|S| > 2$. Here, $1 \in \mathrm{U}(S) + \mathrm{U}(S)$ by Theorem 2.7(2). For $n \geq 2$, consider any nonzero $M = \left(\begin{smallmatrix} A & \beta \\ \gamma & d \end{smallmatrix}\right) \in R$. To prove that $M \in \Phi(R)$, we may assume by Proposition 3.9 that M is in good form; i.e. $A \neq 0$, and $d \neq 0$. By the inductive hypothesis, $A \in \Phi(\mathbb{M}_{n-1}(S))$, and S being a fine ring implies that $d \in \Phi(S)$. By Proposition 3.2, we have then $M \in \Phi(R)$.

We may now assume that $S \cong \mathbb{F}_2$. Indeed, the rest of the proof, again by induction on n, will work *for any division ring S*. To prove that $M = \left(\begin{smallmatrix} A & \beta \\ \gamma & d \end{smallmatrix}\right) \neq 0$ is in $\Phi(R)$, we begin by noting that, by the argument in Case 2 in the proof of Proposition 3.9, we may assume (up to similarity) that $A \neq 0$. If $d \neq 0$, we are done by the argument at the end of the last paragraph. So we may assume that $d = 0$. By the inductive hypothesis, we have a fine decomposition $A = U + T$ in $\mathbb{M}_{n-1}(S)$. Thus, we can write

$$M = \begin{pmatrix} U & \beta + \delta \\ \gamma & 0 \end{pmatrix} + \begin{pmatrix} T & -\delta \\ 0 & 0 \end{pmatrix}, \tag{3.10}$$

where δ is a column vector that is to be chosen. Here, the second summand is nilpotent, so we are done if δ can be chosen such that the first summand on the right-hand side of (3.10) is *invertible.* We go into the following two cases.

Case 1. $\gamma \neq 0$. Following an idea of Kaplansky that was introduced by Henriksen in his paper [16], we perform on the first summand of the right-hand side of (3.10) the following block elementary transformation:

$$\begin{pmatrix} I_{n-1} & 0 \\ -\gamma U^{-1} & 1 \end{pmatrix} \begin{pmatrix} U & \beta + \delta \\ \gamma & 0 \end{pmatrix} = \begin{pmatrix} U & \beta + \delta \\ 0 & -\gamma U^{-1}(\beta + \delta) \end{pmatrix}.$$

In view of this equation, we are done if δ can be chosen such that $\gamma U^{-1}(\beta + \delta) = 1$, for then the first summand on the right-hand side of (3.10) will be invertible. Since $\gamma \neq 0$, we have $\gamma U^{-1} \neq 0$ too, so we can certainly solve the equation $\gamma U^{-1}(\beta + \delta) = 1$ *over the division ring S* since (by varying δ) the vector $\beta + \delta$ can be arbitrarily chosen.

Case 2. $\gamma = 0$. If $\beta \neq 0$, we can repeat the above argument to complete the proof. Thus, we may *also* assume that $\beta = 0$, so now $M = \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Suppose the ith row γ' of A is nonzero. Conjugating M by the elementary matrix $I_n + E_{ni}$, we arrive at a new matrix $\begin{pmatrix} A & 0 \\ \gamma' & 0 \end{pmatrix}$ where $\gamma' \neq 0$. We are now back to Case 1, so the proof is complete. $\square$

For *any* ring R, we have the following important consequence of Theorem 3.1.

Corollary 3.11. *R is a simple artinian ring iff it is a semilocal fine ring.*

Proof. If R is simple artinian, the Wedderburn–Artin theorem [19, (3.5)] implies that $R \cong \mathbb{M}_n(S)$ for some division ring S and some $n \geq 1$. Since S is fine, so is R by Theorem 3.1. Conversely, if R is semilocal and fine, then R is a simple ring by Theorem 2.3(2); in particular, $\mathrm{rad}(R) = 0$. Invoking the definition of a semilocal ring (as in [19, (20.1)]), we see that $R = R/\mathrm{rad}(R)$ is a (simple) artinian ring. $\square$

Remark 3.12. From a linear algebra point of view, the most relevant fact is the "only if" part of Corollary 3.11; namely, that $\mathbb{M}_n(S)$ is a fine ring for any division ring S. In this case, we should observe that, as long as $|S| > 2$, Proposition 3.9 shows that we can conjugate any nonzero matrix $M \in \mathbb{M}_n(S)$ into a matrix N *with all diagonal entries nonzero.* From this, it follows right away that N is the sum of an invertible lower triangular matrix and a strictly upper triangular (and hence nilpotent) matrix.

For certain ring classes $\mathfrak{F}$, it may be true that the fine rings in $\mathfrak{F}$ are all (simple) artinian. An all-too-obvious example of such $\mathfrak{F}$ is the class of reduced rings. A second example of $\mathfrak{F}$ is the class of semilocal rings, according to Corollary 3.11. Yet another example turns out to be the class of endomorphism rings of abelian groups. This is shown by the following result, which depends heavily on the endomorphism theory of abelian groups (as developed, e.g. in Fuchs' book [13]).

Theorem 3.13. *For $R = \mathrm{End}_{\mathbb{Z}}(G)$ where G is any nonzero abelian group, the following four statements are equivalent*:

(1) *R is simple.*
(2) *R is simple artinian.*
(3) *R is a fine ring.*
(4) *G is a finite-dimensional vector space over $\mathbb{Q}$ or over $\mathbb{F}_p$ for some prime p.*

Proof. The equivalence of (1), (2) and (4) is given by [13, Theorem 111.2]. To link these three statements to (3), we simply recall that (3) $\Rightarrow$ (1) is Theorem 2.3(2), and (2) $\Rightarrow$ (3) is the "only if" part of Corollary 3.11. $\square$

The above discussions prompted us to ask the following.

Question 3.14. *What are some examples of classes of exchange rings in which the fine rings are* (*simple*) *artinian*?

More generally, it would be of great interest to determine all fine rings among the exchange rings; hence Question 3.15, part (A) of which is a more specific form of Question 2.10 in view of the result of Camillo and Yu [8, Proposition 10] that a clean ring is 2-good if 2 is invertible.

Question 3.15. *Which simple exchange rings are fine*? Just to name a few concrete special cases, we can ask the following two sub-questions.

(A) Is a simple clean ring R with $2 \in \mathrm{U}(R)$ always fine? (A closely related question would be: which simple unit-regular rings are fine?)
(B) Are there some examples of fine rings among the purely infinite simple rings defined by Ara, Goodearl and Pardo in [3]? (According to Ara [2], the latter rings are always exchange rings.)

4. Corner Rings

This section is devoted to the consideration of the important question *whether the notion of a fine ring is Morita-invariant*; that is, whether the property of being a fine ring is preserved by the Morita equivalence of rings. For the relevant definition of (and facts about) Morita equivalence, see [18, Sec. 18]. Since we did know from Theorem 3.1 that the fineness of rings is preserved by the formation of matrix rings, a standard result in the literature (see, e.g. [18, (18.35)]) implies that the Morita equivalence question above boils down to asking the following.

Question 4.1. *If R is a fine ring and $e \in R$ is a full idempotent* (*that is*, *an idempotent such that $ReR = R$*), *is the corner ring eRe necessarily a fine ring*?

In view of Theorem 2.3(2), such a question is completely in line with the well-known fact that any nonzero corner ring of a simple ring is simple. But Question 4.1 posed for fine rings turns out to be much harder to answer. One may begin by asking more generally how the fine elements of eRe are related to those of R. On this question, one quick thing we can say is the following, even without assuming the idempotent e to be full. Letting f be the complementary idempotent $1 - e$, we have the obvious inclusion relations $\mathrm{U}(eRe) + f \subseteq \mathrm{U}(R)$, and $\mathrm{nil}(eRe) \subseteq \mathrm{nil}(R)$. Together, these two inclusions imply that

$$\Phi(eRe) + f \subseteq \Phi(R). \tag{4.2}$$

However, for a general element $a \in eRe$, $a + f \in \Phi(R)$ may not imply that $a \in \Phi(eRe)$. (This is in sharp contrast to the situation with the sets of unit-regular elements in R and eRe studied by Lam and Murray in [21].) Examples illustrating this will be given in (4.3), (4.5) below, as well as later in (5.8).

While (4.2) does provide a relation between the two fine sets $\Phi(eRe)$ and $\Phi(R)$, it is far from sufficient for answering Question 4.1. One might wonder if *some other* relation between $\Phi(eRe)$ and $\Phi(R)$ may be better suited for getting a positive answer for (4.1). For instance, if it is true that $eRe \cap \Phi(R) \subseteq \Phi(eRe)$ (for any full idempotent $e \in R$), then certainly (4.1) would have a "yes" answer. Unfortunately, this inclusion relation *does not* hold in general, as the following examples (4.3) and (4.5) will show.

Example 4.3. Taking $R = \mathbb{M}_3(\mathbb{Z})$ and $e \in R$ to be the full idempotent $\operatorname{diag}(1,1,0)$, we identify $S := eRe$ with $\mathbb{M}_2(\mathbb{Z})$ (which corresponds to the "2×2 northwest corner" of $\mathbb{M}_3(\mathbb{Z})$). Let $A = \left(\begin{smallmatrix} 1 & 2 \\ -3 & 0 \end{smallmatrix}\right)$, which will be shown to be a *non-fine* element of S (in (5.9) below). As an element of S, A is identified with the 3×3 matrix $\operatorname{diag}(A, 0) \in R$. This matrix turns out to be in $\Phi(R)$ since it has the following fine decomposition:

$$\begin{pmatrix} 1 & 2 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = U + T := \begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{4.4}$$

where $\det(U) = -1$, and $T^3 = 0$. Remarkably, essentially the same construction can be used to show that the 3×3 matrix $\operatorname{diag}(A, 1)$ is *also* in $\Phi(R)$: we can simply use the modified fine decomposition $\operatorname{diag}(A, 1) = U' + T$ where U' is the matrix obtained from U by changing its $(3,3)$-entry from 0 to 1 (with $\det(U') = 1$).

Example 4.5. We will show here that the inclusion relation $eRe \cap \Phi(R) \subseteq \Phi(eRe)$ may fail even in the case where R is a 2×2 matrix ring over some ring S and $e = \operatorname{diag}(1,0)$ (which is a full idempotent of R that is similar to its complementary idempotent). To get such an example, let $S = \mathbb{M}_2(\mathbb{Z})$ as in (4.3). For this choice of S, $R := \mathbb{M}_2(S)$ may be identified with $\mathbb{M}_4(\mathbb{Z})$, and S may then be identified with eRe. Taking the same non-fine matrix $A = \left(\begin{smallmatrix} 1 & 2 \\ -3 & 0 \end{smallmatrix}\right) \in S$ as in (4.3), we can check again that $\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathbb{M}_2(S)$ is a fine matrix in $\mathbb{M}_4(\mathbb{Z})$, with the following fine decomposition:

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U_1 + T_1 := \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 & 1 \\ -3 & 0 & -1 & -2 \\ 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & -1 \end{pmatrix}. \tag{4.6}$$

Here, $\det(U_1) = -1$, and $T_1^4 = 0$. Just as in Example 4.3, essentially the same construction here can be used to show that $\operatorname{diag}(A, 1, 1)$ is also fine in $\mathbb{M}_4(\mathbb{Z})$, with a fine decomposition $U_1' + T_1$ where U_1' is obtained from U_1 by changing its 2×2 southeast corner $\operatorname{diag}(0, 1)$ into $\operatorname{diag}(1, 2)$, with $\det(U_1') = 1$.

For $a \in eRe$, the fact that $a + (1 - e) \in \Phi(R)$ need not imply $a \in \Phi(eRe)$, and that in general $eRe \cap \Phi(R) \nsubseteq \Phi(eRe)$ for full idempotents $e \in R$ should not entirely dash our hopes for a positive answer to Question 4.1. This is because, in working with Question 4.1, we are under the strong assumption that *all* (not just some) elements of $R \backslash \{0\}$ are fine. In Theorem 4.8 below, we will show that

Question 4.1 does have a positive answer in a special case. Its proof will be based on the following observation.

Lemma 4.7. *If $A = (a_{ij})$ is a fine matrix in a matrix ring $R = \mathbb{M}_n(S)$ where S is any ring, then $\sum_{ij} Sa_{ij}S = S$. In particular, if $A = aE_{ij} \in \Phi(R)$ for some $a \in S$ and some matrix unit E_{ij}, then $SaS = S$.*

Proof. It suffices to prove the first conclusion. Let $J = \sum_{ij} S\, a_{ij} S$. Over the factor ring S/J, the zero matrix is the sum of an invertible matrix and a nilpotent matrix. As we have seen already, this implies that $S/J = 0$; that is, $J = S$. □

Note that, in the last conclusion of the lemma above, we can only conclude that $SaS = S$, but not necessarily $a \in \Phi(S)$. This was exactly what Example 4.5 was all about (in the case where $S = \mathbb{M}_2(\mathbb{Z})$ and $n = 2$). Nevertheless, if S is a *commutative* ring, we have the following much sharper result.

Theorem 4.8. *Let $R = \mathbb{M}_n(S)$ where S is a commutative ring. If $a \in S$ is such that $aE_{ij} \in \Phi(R)$ for some i, j, then $a \in \mathrm{U}(S)$. In particular, R is a fine ring iff S is a field.*

Proof. By Lemma 4.7, we have $SaS = S$. Since S is a commutative ring, it follows that $a \in \mathrm{U}(S)$. This gives the "only if" part of the last statement in the theorem. The "if" part follows from Theorem 3.1. □

In connection to the theorem above, there is also the question whether, over any ring S, $a \in \Phi(S)$ would imply that $aE_{ij} \in \Phi(\mathbb{M}_n(S))$ for all i, j. The answer to this question turns out to be "yes". However, in order not to further lengthen this paper, the proof of this fact from [7] will not be presented here.

5. 2 × 2 Matrices: Fine Versus Clean

In this section, we specialize our study of fine elements to the case where $R = \mathbb{M}_2(S)$. In order to use the powerful determinantal criterion for the invertibility of matrices, we shall assume throughout this section that S is a *commutative* ring. The following result gives a simple criterion for a matrix $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in R$ to be fine, and also relates the fineness property of such a matrix to its cleanness property. The formulation of this result is largely inspired by the techniques developed by Khurana and Lam in the paper [17].

Theorem 5.1. *Let $R = \mathbb{M}_2(S)$ where S is a commutative domain. Then $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in R$ is fine iff $b + aS$ contains a unit of S. In this case, A is necessarily a clean matrix in R.*

Proof. First assume that $A \in \Phi(R)$. Then there exists $T \in \mathrm{nil}(R)$ such that $A - T \in \mathrm{U}(R)$. By working over the field of quotients of S, we see that $T^2 = 0$, and

$\det(T) = \mathrm{tr}(T) = 0$. Thus, T has the form $\left(\begin{smallmatrix} s & x \\ y & -s \end{smallmatrix}\right)$ for some $s, x, y \in S$ such that $s^2 + xy = 0$. As $A - T \in R$ is invertible, we have

$$v := \det(A - T) = (a - s)\, s + (b - x)\, y = as + by \in \mathrm{U}(S). \tag{5.2}$$

Note that $y \in \mathrm{U}(S)$. For if otherwise, y would be contained in a maximal ideal $\mathfrak{m}$ of S. Then $s^2 + xy = 0$ implies that $s \in \mathfrak{m}$ too, and (5.2) would give $v \in \mathfrak{m}$, a contradiction. From (5.2), we have thus $b + a\,(sy^{-1}) = vy^{-1} \in \mathrm{U}(S)$.

Conversely, suppose there exists $s \in S$ such that $b + as \in \mathrm{U}(S)$. We complete the proof by essentially reversing the steps in the above work. Taking $y = 1$ and $x = -s^2$, we have $s^2 + xy = 0$, so the matrix $T := \left(\begin{smallmatrix} s & x \\ y & -s \end{smallmatrix}\right)$ satisfies $T^2 = 0$. Also, the computation in (5.2) shows that $\det(A - T) = as + by = as + b \in \mathrm{U}(S)$. Thus, $A - T \in \mathrm{U}(R)$, and hence $A \in \Phi(R)$. (Notice that this part of the proof is valid as long as $S \neq 0$. We do not need to assume here that S is a domain.)

The last statement of the theorem is a special case of [17, Proposition 4.6] which gave a sufficient condition for the *cleanness* of 2×2 matrices with second row zero. To make our arguments completely self-contained, we will give an *ad hoc* proof here. Suppose $b + as \in \mathrm{U}(S)$ for some $s \in S$. To show that A is clean, let E be the idempotent matrix $\left(\begin{smallmatrix} 1+s & -s\,(1+s) \\ 1 & -s \end{smallmatrix}\right) \in R$. Then

$$\det(A - E) = \det\begin{pmatrix} a - 1 - s & b + s + s^2 \\ -1 & s \end{pmatrix} = as + b \in \mathrm{U}(S). \tag{5.3}$$

Thus, $A - E \in \mathrm{U}(R)$, so A is a clean matrix in R, as desired. □

Since $\mathrm{U}(\mathbb{Z}) = \{\pm 1\}$, the following is an immediate consequence of Theorem 5.1.

Corollary 5.4. *A matrix* $A = \left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) \in \mathbb{M}_2(\mathbb{Z})$ *is fine iff* $b \equiv \pm 1 \pmod{a}$. *In this case, A is necessarily a clean matrix in* $\mathbb{M}_2(\mathbb{Z})$.

Next, we note that the following result holds without a domain assumption on S. For the notion of rings of stable range one, see [19, (20.10)].

Corollary 5.5. *If $S \neq 0$ is a commutative ring of stable range one, then* $A = \left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) \in R = \mathbb{M}_2(S)$ *is fine iff* $aS + bS = S$. *In this case, A is necessarily clean in R.*

Proof. The "only if" part follows from Lemma 4.7. For the "if" part, assume that $aS + bS = S$. Since S has stable range one, $b + aS$ contains a unit of S. The second paragraph in the proof of Theorem 5.1 (without any domain assumption on S) now shows that $A \in \Phi(R)$, and the rest of that proof shows that A is clean in R. □

In spite of the last conclusion in Theorem 5.1, it turns out that "fine" and "clean" are independent notions, say for 2×2 matrices over commutative rings. The following example shows that nonzero clean elements in $\mathbb{M}_2(S)$ may not be fine.

Example 5.6. Let $R = \mathbb{M}_2(S)$, where S is a commutative ring. For any non-unit $b \in S$, the nilpotent matrix $A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in R$ is clean.[b] However, since $bS \neq S$, A is *not* fine by Lemma 4.7. Alternatively, we can take a matrix such as $A' = \begin{pmatrix} 5 & 3 \\ 0 & 0 \end{pmatrix}$. This is also clean, with a clean decomposition $\begin{pmatrix} 3 & 2 \\ -3 & -2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$. But A' is *not* a fine matrix in $\mathbb{M}_2(\mathbb{Z})$ according to Corollary 5.4. (Of course, if we are not restricting ourselves to rings of the form $\mathbb{M}_2(S)$, then for any local ring R, all elements of R are clean, but only the units of R can be fine by Proposition 2.2(3).)

To construct a 2×2 matrix that is fine but not clean, we first give a criterion for a diagonal matrix $\mathrm{diag}(a, 1)$ to be clean in $\mathbb{M}_2(S)$.

Theorem 5.7. *Over a commutative domain S, a diagonal matrix $A = \mathrm{diag}(a, 1)$ is clean in $R = \mathbb{M}_2(S)$ iff $a \in \mathrm{U}(S) \cup (1 + \mathrm{U}(S))$.*

Proof. If $a \in \mathrm{U}(S)$, then $A \in \mathrm{U}(R)$ is obviously clean. If $a = 1 + u$ for some $u \in \mathrm{U}(S)$, then $A = \mathrm{diag}(1, 0) + \mathrm{diag}(u, 1)$ is also clean. Conversely, assume that A is clean, and take a clean decomposition $A = E + U \in R$, where $E = E^2$ and $U \in \mathrm{U}(R)$. Clearly, $E \neq I_2$, since $A - I_2 = \mathrm{diag}\,(a - 1, 0) \notin \mathrm{U}(R)$. If $E = 0$, then $A = U \in \mathrm{U}(R)$ implies that $a \in \mathrm{U}(S)$. Assuming now that $E \notin \{0, I_2\}$ (and S is a domain), it is easy to see (e.g. from the Cayley–Hamilton theorem) that E has the form $\begin{pmatrix} s & x \\ y & 1-s \end{pmatrix}$ where $s, x, y \in S$ satisfy $s\,(1 - s) = xy$. Then

$$\det(U) = \det(A - E) = s\,(a - s) - xy = sa - s^2 - xy = s\,(a - 1) \in \mathrm{U}(S)$$

implies that $a - 1 \in \mathrm{U}(S)$, so we have $a \in 1 + \mathrm{U}(S)$. □

Example 5.8. In the setting of Theorem 5.7, if we take any $a \notin \mathrm{U}(S) \cup (1 + \mathrm{U}(S))$, then $A = \mathrm{diag}(a, 1) \in R$ is not clean. However, A is *always fine* (for every $a \in S$), as it has a fine decomposition $\begin{pmatrix} a+1 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$. Thus, for instance, if we choose $a = 3$ or $a = -2$ in $S = \mathbb{Z}$, then the matrix $A = \mathrm{diag}(a, 1)$ is *fine but not clean* in $\mathbb{M}_2(\mathbb{Z})$. Incidentally, this construction provides once more an example of a full idempotent e in a ring R and an element $a \in eRe$ such that $a + (1 - e) \in \Phi(R)$ but $a \notin \Phi(eRe)$.

Example 5.9. The work in our earlier Example 4.3 was based on knowing that a certain integral matrix, namely $\begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix}$, was *not* fine in $R = \mathbb{M}_2(\mathbb{Z})$. We shall now give a full proof for this fact. More generally, we start with a matrix $B = \begin{pmatrix} 1 & b \\ c & 0 \end{pmatrix}$, and first work out the criterion for B to be fine in R. The fineness condition amounts to the existence of a nilpotent matrix $T \in R$ for which $\det(B - T) \in \{\pm 1\}$. Since $\mathbb{Z}$ is an integral domain, we have (as in the proof of Theorem 5.1) $T = \begin{pmatrix} s & x \\ y & -s \end{pmatrix}$ for

[b] In general, any nilpotent element t in any ring is (strongly) clean since it has an obvious (strongly) clean decomposition $t = 1 + (t - 1)$.

some $s, x, y \in \mathbb{Z}$ such that $s^2 + xy = 0$. Thus, the condition for B to be fine can be stated in the form

$$\det \begin{pmatrix} 1-s & b-x \\ c-y & s \end{pmatrix} = s(1-s) - (b-x)\,(c-y) = s - bc + cx + by = \pm 1.$$

Letting $r := bc \pm 1$, we have then $s = r - (cx + by)$, so the equation $s^2 + xy = 0$ becomes

$$c^2x^2 + (2bc+1)\,xy + b^2y^2 - 2crx - 2bry + r^2 = 0. \tag{5.10}$$

Plugging in $(b, c) = (2, -3)$ gives the following two diophantine equations:

$$9x^2 - 11xy + 4y^2 - 30x + 20y + 25 = 0; \tag{5.11}$$

$$9x^2 - 11xy + 4y^2 - 42x + 28y + 49 = 0. \tag{5.12}$$

A quick computer check (using any standard binary quadratic equation solver, such as [1]) shows that neither one of these equations has a solution $(x, y) \in \mathbb{Z}^2$. This proves that the matrix B is indeed non-fine in R. However, B turns out to be *both clean and nil-clean*, according to the following two easy decompositions:

$$B = \begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -3 & 0 \end{pmatrix}. \tag{5.13}$$

Somewhat surprisingly, if we choose $(b, c) = (2, 3)$ instead (changing only the sign of c), then the equation (5.10) has no solution when $r = bc - 1$, but has *exactly two* solutions $(-1, 4)$ and $(9, -16)$ when $r = bc + 1$, according to the computer. What this means is that the corresponding new matrix B_1 has precisely two fine decompositions (in R):

$$B_1 = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} -11 & -7 \\ 19 & 12 \end{pmatrix} + \begin{pmatrix} 12 & 9 \\ -16 & -12 \end{pmatrix}. \tag{5.14}$$

If one is lucky enough to have "guessed" both of these fine decompositions, one is encouraged to try instead $(b, c) = (6, 7)$. In this case, the computer opined that there is a *unique* fine decomposition (in $R = \mathbb{M}_2(\mathbb{Z})$); namely,

$$B_2 = \begin{pmatrix} 1 & 6 \\ 7 & 0 \end{pmatrix} = \begin{pmatrix} -417 & -355 \\ 491 & 418 \end{pmatrix} + \begin{pmatrix} 418 & 361 \\ -484 & -418 \end{pmatrix}. \tag{5.15}$$

This would have (perhaps) defied any guesswork. On the other hand, if we replace the ring $\mathbb{Z}$ by the field $\mathbb{Q}$ and work in $\mathbb{M}_2(\mathbb{Q})$ instead, the matrix B_2 has obviously *infinitely many* fine decompositions, including, for instance,

$$B_2 = \begin{pmatrix} 1 & 6 \\ 7 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 6-q \\ 7 & 0 \end{pmatrix} + \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} \quad (\text{for any } q \in \mathbb{Q}\backslash\{6\}). \tag{5.16}$$

In connection to the work above, we point out that the use of computers in this analysis may lead to new positive results too. For instance, computer calculations indicated that, in *many* cases where $c = b$ or $c = b+2$, the equation (5.10) did have a solution (or solutions) in $\mathbb{Z}$. Based on such empirical data, we find without too much difficulty the following nontrivial fine decompositions (over $\mathbb{Z}$), respectively, for the matrices B_3 and B_4 in the cases where $c = b$ and $c = b+2 = s+1$:

$$B_3 = \begin{pmatrix} 1 & b \\ b & 0 \end{pmatrix} = \begin{pmatrix} -b^2 & -b^2+b-1 \\ b^2+b+1 & b^2+1 \end{pmatrix} + (b^2+1)\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}; \tag{5.17}$$

$$B_4 = \begin{pmatrix} 1 & s-1 \\ s+1 & 0 \end{pmatrix} = \begin{pmatrix} s^2+1 & -s^2+s-1 \\ s^2+s+1 & -s^2 \end{pmatrix} + s^2\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{5.18}$$

Here, on the right-hand side of both equations, the second matrix is nilpotent and the first matrix has determinant 1. Based on (5.17) and (5.18), one might be tempted to conjecture that, in the "intermediate case" $c = b+1$, the matrix $B_5 = \left(\begin{smallmatrix} 1 & b \\ b+1 & 0 \end{smallmatrix}\right)$ might also be fine in $\mathbb{M}_2(\mathbb{Z})$, which we knew to be true at least for $b \in \{2, 6\}$ (by (5.14) and (5.15)). By another computer check using the equation (5.10), one can show that this "conjecture" is true for $b \in \{0, 1, \ldots, 6\}$, but is, unfortunately, false for $b \in \{7, 8\}$.

In the spirit of investigating the relationship between "fineness" and "cleanness", we close the paper with one more query.

Question 5.19. *Is every fine ring necessarily a clean ring*?

Needless to say, the fact that fine elements in rings may not be clean does not preclude this question from having a positive answer. In fact, all examples of fine rings known to the authors have turned out to be clean (or even strongly clean) rings.

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References

[1] D. Alpern, Quadratic equation solver, www.alpertron.com.ar/QUAD.HTM.

[2] P. Ara, The exchange property for purely infinite simple rings, *Proc. Amer. Math. Soc.* **132** (2004) 2543–2547.

[3] P. Ara, K. R. Goodearl and E. Pardo, K_0 of purely infinite simple regular rings, *K-Theory* **26** (2002) 69–100.

[4] N. Ashrafi and P. Vámos, On the unit sum number of some rings, *Quart. J. Math.* **56** (2005) 1–12.

[5] S. Breaz, G. Călugăreanu, P. Danchev and T. Micu, Nil-clean matrix rings, *Linear Algebra Appl.* **439** (2013) 3115–3119.

[6] G. Călugăreanu, UU rings, *Carpathian J. Math.* **31** (2015) 157–163.
[7] G. Călugăreanu and T. Y. Lam, Further notes on fine rings, Unpublished (2015).
[8] V. Camillo and H.-P. Yu, Exchange rings, units and idempotents, *Comm. Algebra* **22** (1994) 4737–4749.
[9] M. Chebotar, P. H. Lee and E. R. Puczylowski, On commutators and nilpotent elements in simple rings, *Bull. London Math. Soc.* **42** (2010) 191–194.
[10] P. Danchev and T. Y. Lam, Rings with unipotent units, *Publ. Math. Debrecen* (2016).
[11] A. J. Diesl, Nil clean rings, *J. Algebra* **383** (2013) 197–211.
[12] N. I. Dubrovin, Rational closures of group rings of left-ordered groups, *Mat. Sb.* **184** (1993) 3–48 (in Russian); *Russian Acad. Sci. Sb. Math.* **79** (1994) 231–263.
[13] L. Fuchs, *Infinite Abelian Groups*, Vol. 2 (Academic Press, New York, 1973).
[14] H. Grover, Z. Wang, D. Khurana, J. Chen and T. Y. Lam, Sums of units in rings, *J. Algebra Appl.* **13** (2014) Article ID: 1350072, 10 pp.
[15] J. Han and W. K. Nicholson, Extensions of clean rings, *Comm. Algebra* **29** (2001) 2589–2595.
[16] M. Henriksen, Two classes of rings generated by their units, *J. Algebra* **31** (1974) 182–193.
[17] D. Khurana and T. Y. Lam, Clean matrices and unit-regular matrices, *J. Algebra* **280** (2004) 683–698.
[18] T. Y. Lam, *Lectures on Modules and Rings*, Graduate Texts in Mathematics, Vol. 189 (Springer-Verlag, Berlin, 1999).
[19] T. Y. Lam, *A First Course in Noncommutative Rings*, 2nd edn. Graduate Texts in Mathematics, Vol. 131 (Springer-Verlag, Berlin, 2001).
[20] T. Y. Lam, *Exercises in Classical Ring Theory*, 2nd edn. Problem Books in Mathematics (Springer-Verlag, Berlin, 2003).
[21] T. Y. Lam and W. Murray, Unit regular elements in corner rings, *Bull. Hong Kong Math. Soc.* **1** (1997) 61–65.
[22] W. K. Nicholson, Lifting idempotents and exchange rings, *Trans. Amer. Math. Soc.* **229** (1977) 269–278.
[23] W. K. Nicholson, Strongly clean rings and Fitting's lemma, *Comm. Algebra* **27** (1999) 3583–3592.
[24] W. K. Nicholson and Y. Zhou, Rings in which elements are uniquely the sum of an idempotent and a unit, *Glasgow Math. J.* **46** (2004) 227–236.
[25] P. Vámos, 2-good rings, *Quart. J. Math.* **56** (2005) 417–430.
[26] D. Zelinsky, Every linear transformation is a sum of two nonsingular ones, *Proc. Amer. Math. Soc.* **5** (1954) 627–630.