Example revisited

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In [2], the matrix $C = \begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix} \in \mathcal{M}_2(\mathbf{Z})$ was given as an example of nil-clean matrix which is not clean, using a long, fairly difficult process, involving solving Diofantine equations. Meanwhile, the second author "discovered" that Diofantine equations may be (instantly) solved using computer aid, that is using [1] (which was there from 2001 !).

In the sequel we revisit this example using [1]. A (nil-)clean element is called *trivial (nil-)clean* if its decomposition uses a trivial idempotent (i.e. 0 or 1).

It is easy to prove the following two results (see also [3], [4], [5]).

Theorem 1 *A 2×2 integral matrix A is nontrivial nil-clean iff A has the form $\begin{bmatrix} a+1 & b \\ c & -a \end{bmatrix}$ for some integers a, b, c such that $\det(A) \neq 0$ and the system*

$$\begin{cases} x^2 + x + yz = 0 & (1) \\ (2a+1)x + cy + bz = a^2 + bc & (2) \end{cases}$$

with unknowns x, y, z, has at least one solution over $\mathbf{Z}$. We can suppose $b \neq 0$ and if (2) holds, (1) is equivalent to

$$bx^2 - (2a+1)xy - cy^2 + bx + (a^2 + bc)y = 0 \qquad (3).$$

Theorem 2 *A 2×2 integral matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nontrivial clean iff the system*

$$\begin{cases} x^2 + x + yz = 0 & (1) \\ (a-d)x + cy + bz + \det(A) - d = \pm 1 & (\pm 2) \end{cases}$$

with unknowns x, y, z, has at least one solution over $\mathbf{Z}$. *If* $b \neq 0$ *and (2) holds, then (1) is equivalent to*

$$bx^2 - (a-d)xy - cy^2 + bx + (d - \det(A) \pm 1)y = 0 \qquad (\pm 3).$$

Since C is not nilpotent nor unipotent ($1+t$ with nilpotent t), it is not trivial nil-clean.

Using the first theorem, for $a = 2, b = 9, c = -7$ we have $9x^2 - 5xy + 7y^2 + 9x - 59y = 0$ (3), with solutions $(0,0)$, $(-1,0)$ and $(2,9)$. Since in $5x - 7y + 9z = -59$ (2) there is an integer solution for z only for $(x,y) = (-1,0)$, the matrix is (even) *uniquely nil-clean* with decomposition

$$\begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -6 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}.$$

Next, since C and $C - I_2$ are not units, it follows that C is not trivial clean.

Using the second theorem, for $a = 3$, $b = 9$ $c = -7$ and $d = -2$ we have $9x^2 - 5xy + 7y^2 + 9x + (-59 \pm 1)y = 0$ (± 3), with solutions $(0,0)$, $(-1,0)$ respectively $(0,0)$, $(-1,0)$, $(3,9)$ and $(1,9)$. Now (± 2) are $5x - 7y + 9z + 59 = \pm 1 \quad (\pm 2)$ and one can check that the solutions of (+3) do not verify (+2), and the solutions of (-3) do not verify (-2). Hence C is (indeed) not clean.

References

[1] D. Alpern *Quadratic equation solver.* www.alpertron.com.ar/QUAD.HTM.

[2] D. Andrica, G. Călugăreanu *A nil-clean 2 x 2 matrix over integers which is not clean.* J. of Algebra and its Appl., vol. **13**, 6 (2014), 9 pages.

[3] D. Andrica, G. Călugăreanu *Uniquely clean 2 x 2 invertible integral matrices.* to appear in Studia Universitatis Babeş-Bolyai (2017)

[4] G. Călugăreanu *Nil-clean integral 2 x 2 matrices; The elliptic case.* to appear

[5] G. Călugăreanu *Clean integral 2 x 2 matrices.* to appear