2-PRODUCTS OF IDEMPOTENT BY NILPOTENT MATRICES

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Abstract. Over Prüfer domains we characterize idempotent by nilpotent 2-products of 2×2 matrices. Nilpotents are always such products. We also provide large classes of rings over which every 2×2 idempotent matrix is such a product. Finally, for 2×2 matrices over GCD domains, idempotent-nilpotent products which are also nilpotent-idempotent products are characterized.

1. Introduction

In any unital ring, the set of units, the set of nilpotents and the set of idempotents are of utmost importance. By taking sums of elements in these sets, *clean* elements (sums of units and idempotents), *nil-clean* elements (sums of nilpotents and idempotents) and *fine* elements (sums of units and nilpotents) were defined (clean in [10], nil-clean in [7] and fine in [4]) and accordingly clean (or nil-clean or fine) rings were defined and studied in depth.

There is less literature on products of elements in these sets. Products of units and idempotents were studied by G. Ehrlich (see [8] and [9]) defining the so-called *unit-regular* rings, products of units and nilpotents were studied in [2] and [12] defining the so-called *UN-rings*, and rings for which every nonunit is a product of a nilpotent and an idempotent (in either order) were recently characterized in [14]. As a multiplicative dual of a nil-clean element, an element a of a ring R was called *dual nil-clean* if $a = et$ where e is an idempotent and t is a nilpotent.

In this note we study the set of products et with idempotent e and nilpotent t, mainly for 2×2 matrices over Prüfer domains. Such products will be called shortly *IN-elements*. *NI-elements* (i.e., te) are defined symmetrically.

Since 1 is idempotent, in every unital ring *nilpotents are IN and NI-elements, but no unit is an IN or NI-element* (indeed, for an idempotent e, a nilpotent t and a unit u, $et = u$ implies $et = eu = u$ so $e = 1$ and $t = u$, a contradiction). Note that if $a \in R$ is IN (or NI) then all conjugates of a are also IN (resp. NI). As for matrices, there conditions are invariant to similarities.

As a remaining possibility, we focus on *idempotents which are IN-elements* (see section 3). Clearly, we restrict to nontrivial idempotents since 0 is IN and NI and 1 is not.

Since our characterization of IN-matrices is over Prüfer domains (see Section 2), recall that a *Prüfer domain* is a semihereditary integral domain (i.e., a ring R is called *semihereditary* if all finitely generated submodules of projective modules over R are again projective). Equivalently, an integral domain R is Prüfer if every

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nonzero finitely generated ideal of R is invertible (an ideal I is *invertible* if $I \cdot I^{-1} = R$ where $I^{-1} = \{r \in q(R) : rI \subseteq R\}$ and $q(R)$ is the field of fractions of R). Fields, PIDs and Bézout domains are Prüfer domains but unique factorization domains may not be Prüfer.

In section 3 we emphasize some large classes of rings (namely the GCD - greatest common divisors exist - domains) over which all 2×2 idempotents have IN-decompositions. We also show that this result does not generalize for $n \times n$ idempotent matrices if $n > 2$. By $E_{ij} \in \mathbb{M}_n(R)$ we denote the standard matrix unit, i.e., E_{ij} has 1 as (i, j)-entry and zeros elsewhere.

Notice that in any ring, every idempotent-unit product is also a unit-idempotent product and every nilpotent-unit product is also a unit-nilpotent product. Indeed, if au is such a product, with idempotent or nilpotent a and unit u, then $au = u(u^{-1}au)$ and $u^{-1}au$, as conjugate of a, is also idempotent or nilpotent, respectively.

In closing, over GCD domains we characterize the IN 2×2 matrices which are also NI, give several examples and an application of our results. Especially in this last characterization, computer guidance was essential.

2. IN and NI-matrices

In our characterization theorem we intend to use the Kronecker (Rouché) - Capelli theorem for compatible linear systems. As early as 1971 we recall from [5] the following characterization.

Theorem 1. *Let R be an integral domain, A a matrix of rank r over R and $\mathbf{x}$ and $\mathbf{b}$ column vectors over R. The condition $D_r(A) = D_r[A, \mathbf{b}]$ is necessary and sufficient for the system $A\mathbf{x} = \mathbf{b}$ to be solvable iff R is a Prüfer domain.*

Here the ideal $D_t(A)$ generated by the $t \times t$ minors of the matrix is called the *t-th determinantal ideal* of A and we put $D_0(A) = 1$. As customarily, $[A, \mathbf{b}]$ denotes the augmented matrix. Also recall (e.g., see [1]) that, over an arbitrary commutative ring, the rank of an $n \times n$ matrix is the following integer $rk(A) = \max\{t : Ann_R(D_t(A)) = (0)\}$. Moreover, over integral (commutative) domains, the rank agrees with the classical definition of rank, that is, the largest integer k such that the matrix contains a $k \times k$ submatrix whose determinant is nonzero.

Theorem 2. *A 2×2 zero determinant matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ over a Prüfer domain R is IN iff there exist $a, b, c \in R$ with $bc = a(1-a)$ such that $crow_1(A) = arow_2(A)$ and a_{11}^2, a_{12}^2 and $a_{11}a_{12}$ are divisible by $ba_{11} - aa_{12}$. The divisibilities are equivalent with a_{21}^2, a_{22}^2 and $a_{21}a_{22}$ being divisible by $(1-a)a_{21} - ca_{22}$.*

We discuss separately the cases $a \in \{0, 1\}$, so below we assume $a, b, c \neq 0$ and $a \neq 1$.

Proof. It is well-known that over any integral domain a 2×2 IN-matrix is of form $ET = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$, with $a(1-a) = bc$ and $x^2 + yz = 0$. Denoting $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the equality $A = ET$ amounts to the linear system in unknowns

x, y, z

$$\begin{array}{lcl} ax+bz & = & a_{11} \\ -bx+ay & = & a_{12} \\ cx+(1-a)z & = & a_{21} \\ -(1-a)x+cy & = & a_{22} \\ a(1-a) & = & bc \\ x^2+yz & = & 0 \end{array}.$$

The first four equations form a linear system with 3 unknowns and 4 equations whose augmented matrix is $\begin{bmatrix} a & 0 & b & a_{11} \\ -b & a & 0 & a_{12} \\ c & 0 & 1-a & a_{21} \\ a-1 & c & 0 & a_{22} \end{bmatrix}$.

An easy computation shows that the system matrix $\begin{bmatrix} a & 0 & b \\ -b & a & 0 \\ c & 0 & 1-a \\ a-1 & c & 0 \end{bmatrix}$ has rank 2, as $a(1-a) = bc$.

According to the previous theorem, since we work over a Prüfer domain, the solvability of this system amounts to the equality of the ranks of the 4×3 system matrix and the augmented 4×4 matrix. As the 3×3 minors of the system matrix are zero, so is the determinant of the augmented 4×4 matrix. Another easy computation shows that the remaining twelve 3×3 minors of the augmented matrix are zero iff $crow_1(A) = arow_2(A)$, so this is the necessary and sufficient condition for the equality of the ranks, that is, for the solvability of the system.

Thus, in order to find a solution we select (say) the first two equations i.e., $ax+bz = a_{11}$, $-bx+ay = a_{12}$. The existence of this solution requires the divisibilities in the statement. Next, to simplify the writing, we *formally* use fractions. Then (temporary) $x = \dfrac{a_{11}-bz}{a}$ and $y = \dfrac{b(a_{11}-bz)+aa_{12}}{a^2}$ and replacing in $x^2+yz = 0$ we get $x = -\dfrac{-a_{11}a_{12}}{ba_{11}-aa_{12}}$, $y = -\dfrac{a_{12}^2}{ba_{11}-aa_{12}}$ and $z = \dfrac{a_{11}^2}{ba_{11}-aa_{12}}$. As required in the statement, for instance, since $ba_{11}-aa_{12}$ divides $a_{11}a_{22}$, there exists x for a solution. Finally, the IN-decomposition is the following

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} \begin{bmatrix} \dfrac{-a_{11}a_{12}}{ba_{11}-aa_{12}} & -\dfrac{a_{12}^2}{ba_{11}-aa_{12}} \\ \dfrac{a_{11}^2}{ba_{11}-aa_{12}} & -\dfrac{-a_{11}a_{12}}{ba_{11}-aa_{12}} \end{bmatrix}$$

with $a(1-a) = bc$. □

Remarks. 1) Since the rows of A are dependent, clearly $\det(A) = 0$. But this follows at once since $\det(ET) = \det(E)\det(T) = 0\cdot 0 = 0$.

2) **The case** $a = 1$. As $a(1-a) = bc$, at least one of b, c must be zero and (say $c = 0$) $E = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$. For a matrix $A = [a_{ij}]_{1\le i,j\le 2}$ to have such an IN-decomposition, $a_{21} = a_{22} = 0$ are necessary conditions. As in the previous proof, $x = a_{11}-bz$, $y = a_{12}+bx = a_{12}+b(a_{11}-bz)$ and $x^2+yz = 0$ gives $x = -\dfrac{-a_{11}a_{12}}{ba_{11}-a_{12}}$, $y = -\dfrac{a_{12}^2}{ba_{11}-a_{12}}$ and $z = \dfrac{a_{11}^2}{ba_{11}-a_{12}}$ with a_{11}^2, a_{12}^2 and $a_{11}a_{12}$ divisible by $ba_{11}-a_{12}$.

The case $b = 0$ follows by transpose.

The case $a = 0$. Again at least one of b, c must be zero and (say $c = 0$) $E = \begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix}$. The first two equations of the linear system are $bz = a_{11}$, $-bx = a_{12}$. Therefore both a_{11}, a_{12} are divisible by b, $x = -\dfrac{a_{12}}{b}$, $z = \dfrac{a_{11}}{b}$ and $y = -\dfrac{a_{12}^2}{ba_{11}}$, which requires a_{12}^2 being divisible by a_{11}.

Examples. 1) Consider $A = E_{11} \in \mathbb{M}_2(R)$, i.e., $a_{11} = 1$, $a_{12} = a_{21} = a_{22} = 0$. Since $ba_{21} = (1-a)a_{11}$ it follows that $a = 1$ and as $ca_{11} = aa_{21}$, $c = 0$. Finally, for a_{11}, a_{12} to be divisible by $ba_{11} - aa_{12}$ we choose $b = 1$. Thus $E_{11} = (E_{11} + E_{12})E_{21}$ shows E_{11} is IN *over any ring*.

2) Consider $A = \begin{bmatrix} 1 & \gamma \\ 0 & 0 \end{bmatrix}$. Since $c \begin{bmatrix} 1 & \gamma \end{bmatrix} = a \begin{bmatrix} 0 & 0 \end{bmatrix}$ it follows that $c = 0$. Thus $a = 0$ or $a = 1$ and it is readily checked that $a = 0$ is not suitable. For $a = 1$, $b - \gamma$ should divide 1, γ^2 and γ so we can take $b = 1 + \gamma$. Searching for $\begin{bmatrix} 1 & \gamma \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1+\gamma \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$ with $x^2 + yz = 0$, we finally find the IN-decomposition $\begin{bmatrix} 1 & \gamma \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1+\gamma \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\gamma & -\gamma^2 \\ 1 & \gamma \end{bmatrix}$. The other idempotents $\begin{bmatrix} 1 & 0 \\ \gamma & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ \gamma & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & \gamma \\ 0 & 1 \end{bmatrix}$ have analogous IN-decompositions.

Alternative proof: Recall that two square matrices A, B of the same size are called *equivalent* if there are invertible matrices U, V (of the same size) such that $B = UAV$ and *similar* if $V = u^{-1}$. In either case, it is easy to see that these (nontrivial) idempotents are equivalent to E_{11}. Hence, by [11], these are also similar to E_{11}, and so IN, by Example 1) above.

We can generalize example 1 as follows

Proposition 3. *Let* $e = e^2 \in R$. *The (nontrivial)* $n \times n$ *idempotent* eE_{11} *is IN over any (unital) ring.*

Proof. Indeed, $eE_{11} = e(E_{11} + E_{12} + ... + E_{1n})E_{n1}$ is an IN decomposition over any ring. □

Corollary 4. *Any* $n \times n$ *idempotent similar to* eE_{11} *is IN, over any (unital) ring.*

Corollary 5. *All integral* 2×2 *nontrivial idempotents are IN.*

Proof. Over $\mathbb{Z}$, every nontrivial idempotent is similar with E_{11} (see also last section) So it suffices to check E_{11} for IN. □

The characterization is easier to check over a pre-Schreier domain, since zero determinant 2×2 matrices are *non-full*, that is, decompose in a column-row product (see [3]). Recall that a commutative ring R is called *pre-Schreier*, if every nonzero element $r \in R$ is *primal*, i.e., if r divides xy, there are r_1, r_2 elements in R such that $r = r_1 r_2$, r_1 divides x and r_2 divides y. Pre-Schreier domains were introduced by M. Zafrullah in [13].

A pre-Schreier integrally closed domain was called a *Schreier* domain by P. M. Cohn in [6]. Every GCD domain is Schreier.

Theorem 6. *Let* $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \begin{bmatrix} \gamma & \delta \end{bmatrix}$ *be a (zero determinant)* 2×2 *matrix over a pre-Schreier domain* R. *Then* A *is IN iff there exist* $a, b, c \in R$ *such that* $bc = a(1-a)$, $c\alpha = a\beta$ *and all* $\alpha\gamma^2, \alpha\delta^2$ *and* $\alpha\gamma\delta$ *are divisible by* $b\gamma - a\delta$.

Example. The zero determinant integral matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has no IN-decomposition.

Indeed, as in the proof of Theorem 2, for a solution we have to solve $ax + bz = 1$, $-bx + ay = 2$. Replacing in $x^2 + yz = 0$ gives $x = \dfrac{-2}{b-2a}$, $y = \dfrac{-4}{b-2a}$ and $z = \dfrac{1}{b-2a}$. Over $\mathbb{Z}$ we must have $b - 2a \in \{\pm 1\}$ and so $T = \begin{bmatrix} \mp 2 & \mp 4 \\ \pm 1 & \pm 2 \end{bmatrix}$. The idempotent E satisfies $b - 2a \in \{\pm 1\}$ and $bc = a(1-a)$, that is, a, c satisfy the quadratic Diophantine equation $a(1-a) = (2a \pm 1)c$. The $+$ equation has the solutions $(0,0)$, $(-1,0)$ and the - equation has also these solutions and two more: $(2,-2)$, $(1,-2)$. None of the four resulting idempotent matrices satisfies $A = ET$.

Alternative proof. Using Theorem 6, $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}$ so $\alpha = 1 = \gamma$, $\beta = 2 = \delta$, $c = 2a$ and $bc = a(1-a)$ whence $a + 2b = 1$ or $a = 0$. Moreover $\alpha\gamma^2 = 1$, $\alpha\delta^2 = 4$, $\alpha\gamma\delta = 2$ should be divisible by $b - 2a$.

If $a = 0$ then $c = 0$ and only $b = \pm 1$ verifies the divisibilities. However $ET = \begin{bmatrix} 0 & \pm 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ z & -x \end{bmatrix} = \begin{bmatrix} \pm z & \mp x \\ z & -x \end{bmatrix} \neq A$.

If $a + 2b = 1$, then $a = 1 - 2b$ and $b - 2a = b - 2(1-2b) = 5b - 2 \neq \pm 1$, so does not divide 1, 4, 2.

Obviously a matrix is IN iff its transpose is NI.

A characterization for NI 2×2 matrices can be proved in an analogous way to Theorem 2, or else, it follows by transpose. Here is the corresponding statement.

Theorem 7. *A* 2×2 *zero determinant matrix* $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ *over a Prüfer domain is NI iff there exist* a, b, c *with* $bc = a(1-a)$ *such that* $b\,col_1(A) = a\,col_2(A)$ *and* a_{11}^2, a_{21}^2 *and* $a_{11}a_{21}$ *are divisible by* $ca_{11} - aa_{21}$.

The case $a = 0$. At least one of b, c must be zero (say $b = 0$) and so $E = \begin{bmatrix} 0 & 0 \\ c & 1 \end{bmatrix}$.

The first and third equations of the linear system are $cy = a_{11}$, $-cx = a_{21}$. Therefore both a_{11}, a_{21} are divisible by c, $x = -\dfrac{a_{21}}{c}$, $y = \dfrac{a_{11}}{c}$ and $z = -\dfrac{a_{21}^2}{ca_{11}}$, which requires a_{21}^2 being divisible by a_{11}.

Special case. If $b = a = 0$ and $c = 1$ the conditions are fulfilled whenever a_{21}^2 is divisible by a_{11}.

Over pre-Schreier domains, we obtain an analogue to Theorem 6.

Theorem 8. *Let* $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \begin{bmatrix} \gamma & \delta \end{bmatrix}$ *be a (zero determinant)* 2×2 *matrix over a pre-Schreier domain* R. *Then* A *is NI iff there exist* $a, b, c \in R$ *such that* $bc = a(1-a)$, $b\gamma = a\delta$ *and all* $\alpha^2\gamma$, $\beta^2\gamma$ *and* $\alpha\beta\gamma$ *are divisible by* $c\alpha - a\beta$.

It is easy to check that analogous results with respect to E_{11}, as those previously proved for IN matrices, can be stated and proved for NI matrices.

3. Idempotent IN-matrices

In this section we largely generalize Corollary 5.

An integral domain is a *GCD* domain if every pair a, b of nonzero elements has a greatest common divisor, denoted by $\gcd(a,b)$. GCD domains include unique factorization domains, Bézout domains and valuation domains. A basic property of a GCD domain is needed for the next proposition: if a divides bc and $\gcd(a,b)=1$, then a divides c.

Proposition 9. *Let R be a GCD domain. Then every nontrivial idempotent is similar to E_{11}.*

Proof. Let $E=\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ be a nontrivial idempotent, i.e. $bc=a(1-a)$ and let $x=\gcd(a,c)$. If $a=xy$ and $c=xx'$ it follows that $\gcd(y,x')=1$. Since $bx'=y(1-a)$, by the GCD hypothesis, y divides b, say $b=yy'$. Now take $P=\begin{bmatrix} x & y' \\ -x' & y \end{bmatrix}$. One can check that $\det(P)=1$ (the case $a=0$ was previously discussed; if $a\neq 0$, one checks $a\det(P)=a$) and $PE=E_{11}P$. Hence E is similar to E_{11}. □

It follows that

Corollary 10. *Over any GCD domain, all idempotent 2×2 matrices are IN.*

The following result emphasizes a 3×3 idempotent matrix which is not IN. A ring R is *connected* if its only idempotents are $0,1$.

Proposition 11. *Over any connected commutative ring R with char$(R)\neq 2$, the idempotent $E_{11}+E_{22}$ is not IN in $\mathbb{M}_3(R)$.*

Proof. Suppose $E_{11}+E_{22}=ET$ for a 3×3 idempotent E and a 3×3 nilpotent T. By left multiplication with E, we get $E(E_{11}+E_{22})=E_{11}+E_{22}$.

Denote $E=[e_{ij}]$, $1\leq i,j\leq 3$. From the last equality it follows that $E=\begin{bmatrix} 1 & 0 & e_{13} \\ 0 & 1 & e_{23} \\ 0 & 0 & e_{33} \end{bmatrix}$ with $\det(E)=e_{33}$. In order E to be an idempotent (but not a unit, recall that R is connected) $e_{33}=0$ is necessary (actually $e_{13}e_{33}=e_{23}e_{33}=0$ and $e_{33}^2=e_{33}$; for $e_{33}=1$, E is a unit) and it turns out to be also sufficient. To simplify the writing denote $e_{13}=b$, $e_{23}=c$. We are now searching for a nilpotent $T=[t_{ij}]$, $1\leq i,j\leq 3$ such that $E_{11}+E_{22}=\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 0 \end{bmatrix}T$. By computation, $T=\begin{bmatrix} 1-bt_{31} & -bt_{32} & -bt_{33} \\ -ct_{31} & 1-ct_{32} & -ctl_{33} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}$. Adding c times the third row to the second row, it is readily seen that $\det(T)=t_{33}$ so $t_{33}=0$ is necessary. Further $Tr(T)=2-bt_{31}-ct_{32}=0$ and $Tr(T^2)=(1-bt_{31})^2+2bct_{31}t_{32}+(1-ct_{32})^2=0$. Replacing $bt_{31}=2-ct_{32}$ in the last equality gives $2=0$, impossible. □

Corollary 12. *Over any connected commutative ring R with $char(R) \neq 2$, any (nontrivial) idempotent 3×3 matrix similar to $E_{11} + E_{22}$ is not IN.*

Hence we cannot generalize Corollary 10 for positive integers $n > 2$.

4. IN 2×2 matrices that are NI

Note that according to Zhou theorem of characterization (Theorem 2.3 in [14]), not only nilpotent and idempotent 2×2 matrices are IN, but all not invertible ones are IN, whenever the base ring is a division ring (commutativity not assumed).

Moreover, note that not every IN 2×2 matrix is also NI. The example of IN matrix which is not NI given in [14] (see Example 2.5), is not in a full matrix ring but (understandable) in a not symmetric subring of $\mathbb{M}_2(\mathbb{Z})$, namely $A = \begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{bmatrix}$. Notice that if we consider $A \in \mathbb{M}_2(\mathbb{Z})$, then $A = E_{11} \begin{bmatrix} -4 & -2 \\ 8 & 4 \end{bmatrix} = 2E_{12} \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$ is IN and also NI (we can use Theorem 7 for NI).

In what follows, *over GCD domains,* we provide a characterization of the IN matrices which are (or are not) NI.

We start with some useful reductions.

First suppose $A = ET$ is IN. By invariance to similarity and Proposition 9, it suffices to characterize the IN-matrices whose idempotent is $E = E_{11}$. Thus we have to find matrices of form $A = E_{11}T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$, with $\beta \mid \alpha^2$, which have (or have not) a NI-decomposition $SF = \begin{bmatrix} x & y \\ z & -x \end{bmatrix} \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ with $x^2 + yz = 0$ and $a(1-a) = bc$.

A **second** reduction involves the possibilities $b = 0$ or $c = 0$.

Notice that $a \in \{0, 1\}$ iff $b = 0$ or $c = 0$ and we distinguish two cases.

If $b = 0$, the idempotent is of form $\begin{bmatrix} 0 & 0 \\ c & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ c & 0 \end{bmatrix}$. Accordingly, $SF = \begin{bmatrix} cy & y \\ -cx & -x \end{bmatrix}$ or $SF = \begin{bmatrix} x + cy & 0 \\ -cx + z & 0 \end{bmatrix}$. The second subcase requires $\beta = 0$ and since $\beta \mid \alpha^2$, also $\alpha = 0$, a trivial case. The first subcase requires $y = \beta$, $x = 0$ and so $\alpha = c\beta$, that is, β not only divides α^2 but divides also α.

In this case, an NI-decomposition is $A = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \frac{\alpha}{\beta} & 1 \end{bmatrix}$.

If $c = 0$, the idempotent is of form $\begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$. Accordingly, $SF = \begin{bmatrix} 0 & bx + y \\ 0 & -x + bz \end{bmatrix}$ or $SF = \begin{bmatrix} x & bx \\ z & bz \end{bmatrix}$. The first subcase requires $\alpha = 0$, so A is nilpotent with the obvious NI-decomposition $(\beta E_{12})I_2$. The second subcase requires $x = \alpha$, $z = 0$ and $bx = \beta$, whence $\beta = b\alpha$, that is, α divides β. In this case, an NI-decomposition is $A = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1-\beta & \dfrac{\beta(1-\beta)}{\alpha} \\ \alpha & \beta \end{bmatrix}$.

A **third** reduction involves signs, and so is useful mainly over $\mathbb{Z}$.

Lemma 13. *If a matrix* $\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$ *is NI, so are* $\begin{bmatrix} -\alpha & \beta \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} \alpha & -\beta \\ 0 & 0 \end{bmatrix}$ *and* $\begin{bmatrix} -\alpha & -\beta \\ 0 & 0 \end{bmatrix}$.

Proof. Assume $\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$. Then

$\begin{bmatrix} -\alpha & \beta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -x & y \\ z & x \end{bmatrix}\begin{bmatrix} a & -b \\ -c & 1-a \end{bmatrix}$,

$\begin{bmatrix} \alpha & -\beta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x & -y \\ -z & -x \end{bmatrix}\begin{bmatrix} a & -b \\ -c & 1-a \end{bmatrix}$ and

$\begin{bmatrix} -\alpha & -\beta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -x & -y \\ -z & x \end{bmatrix}\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$. □

Accordingly, over $\mathbb{Z}$, we can deal only with the $\alpha, \beta \geq 0$, 2×2 matrices with zero second row.

Next a simple but key result.

Proposition 14. *If* $\beta \nmid \alpha$ *then the nilpotent in any NI-decomposition of* $\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$ *is a multiple of* E_{12}.

Proof. Let $\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ with $\beta \nmid \alpha$, $a(1-a) = bc$ and $x^2 + yz = 0$. The matrix equality amounts to four equalities: $-cx + az = 0 = (a-1)x + bz$ and $ax + cy = \alpha$, $bx + (1-a)y = \beta$.

If $z \neq 0$ we multiply $ax + cy = \alpha$ by z. Using $cx = az$ we get $c(x^2 + yz) = \alpha z$ and so $\alpha = 0$, a contradiction, as $\beta \nmid \alpha$. Hence $z = 0$ and so $x = 0$. □

Now we are ready to prove our characterization.

Theorem 15. *Over any GCD domain an IN-matrix* $\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$ *is (also) NI iff there exists a common divisor* δ *of* α *and* β *such that if* $\alpha = \delta\alpha_1$, $\beta = \delta\beta_1$, α_1 *divides* $(\beta_1 - 1)\beta_1$ *or* $\beta_1(\beta_1 + 1)$.

Proof. Owing to our second reduction above, our hypotheses are $\beta \mid \alpha^2$, $\beta \nmid \alpha$ and $\alpha \nmid \beta$. According to the above proposition, in the given hypotheses, the NI-decomposition has to be of form $\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \delta \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1-\beta_1 & * \\ \alpha_1 & \beta_1 \end{bmatrix}$ or else $\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\delta \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1+\beta_1 & * \\ -\alpha_1 & -\beta_1 \end{bmatrix}$. Since the RHS idempotent should have zero determinant (not only trace = 1), α_1 must divide a product $(\beta_1 - 1)\beta_1$ or $\beta_1(\beta_1 + 1)$, respectively, in order to be able to complete $\begin{bmatrix} 1-\beta_1 & * \\ \alpha_1 & \beta_1 \end{bmatrix}$ or $\begin{bmatrix} 1+\beta_1 & * \\ -\alpha_1 & -\beta_1 \end{bmatrix}$ to zero determinant matrices. □

Two special cases are worth mentioning.

1. For $\delta = 1$, α must divide $(\beta - 1)\beta$ or $\beta(\beta + 1)$. In particular, this holds if α divides any of $\beta - 1$, β or $\beta + 1$.

2. For $\delta = \gcd(\alpha, \beta)$, if now $\alpha = \delta\alpha'$, $\beta = \delta\beta'$, since $\gcd(\alpha', \beta') = 1$, α' must divide $\beta' - 1$ or $\beta' + 1$.

4.1. **Examples.** If R is any commutative ring and $r, s \in R$, the following are explicit examples of IN matrices that are also NI.

$$\begin{bmatrix} 0 & r \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ s & 0 \end{bmatrix} = \begin{bmatrix} rs & 0 \\ s & 0 \end{bmatrix} = \begin{bmatrix} rs & r^2 s \\ s & rs \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\begin{aligned} \begin{bmatrix} -r & -r \\ r+1 & r+1 \end{bmatrix}\begin{bmatrix} 0 & -r-1 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & r(r+1) \\ 0 & -(r+1)^2 \end{bmatrix} = \\ &= \begin{bmatrix} -r(r+1)^2 & -r^2(r+1) \\ (r+1)^3 & r(r+1)^2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} -r & 1 \\ -r(r+1) & r+1 \end{bmatrix}\begin{bmatrix} 0 & -r-1 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & r(r+1) \\ 0 & r(r+1)^2 \end{bmatrix} = \\ &= \begin{bmatrix} -r(r+1)^2 & r(r+1) \\ -r(r+1) & r(r+1)^2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} -r & r \\ -r-1 & r+1 \end{bmatrix}\begin{bmatrix} 0 & -r-1 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & r(r+1) \\ 0 & (r+1)^2 \end{bmatrix} = \\ &= \begin{bmatrix} r(r+1)^2 & -r^2(r+1) \\ (r+1)^3 & -r(r+1)^2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Finally, a discussion for some special integral IN matrices whose idempotent is E_{11}.

Proposition 16. *The integral IN-matrices* $A_n = \begin{bmatrix} 2 \times (2n+1) & 2^2 \\ 0 & 0 \end{bmatrix} =$
$= E_{11} \begin{bmatrix} 2 \times (2n+1) & 2^2 \\ (2n+1)^2 & -2 \times (2n+1) \end{bmatrix}$, $n \in \mathbb{Z}$, *are also NI only for* $n \in \{-3, -2, -1, 0, 1, 2\}$.

Proof. The equality $A_n = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ amounts to two systems: $ax + cy = 2(2n+1)$, $bx + (1-a)y = 4$ and $-cx + az = 0$, $-(1-a)x + bz = 0$.

Since $2(2n+1) \nmid 4$ and $4 \nmid 2(2n+1)$, as previously noticed, we can assume $b, c \neq 0$ and so $a \notin \{0, 1\}$.

Both systems have zero determinant (indeed, $a(1-a) = bc$) so the first has solutions only if $\Delta_x = \Delta_y = 0$, that is, $(2n+1)(1-a) = 2c$ and $2a = (2n+1)b$.

As $2 \mid (2n+1)b$ we have $b = 2k$ and so $a = (2n+1)k$ for some k. Further, $2c = (2n+1)[1-(2n+1)k]$ requires odd k, say $k = 2l+1$ whence $c = -(2n+1)(2nl+n+l)$.

In these conditions each system reduces to only one equation, namely $(2l+1)x = (2nl+n+l)y + 2$ and $(2l+1)z = -(2nl+n+l)x$. By computation $(2l+1)^2 z = -(2nl+n+l)[(2nl+n+l)y+2]$ and so $0 = (2l+1)^2(x^2+yz) = 2[(2nl+n+l)y+2]$. However, if $n \geq 3$ then $(2nl+n+l)y = -2$ has no integer solution. Indeed, the quadratic (Diophantine) equations $2nl + n + l \in \{\pm 1, \pm 2\}$ have solutions only for n (or l) in $\{-3, -2, -1, 0, 1, 2\}$.

The NI-decomposition for $n = 1, 2$ are $\begin{bmatrix} 2\times 3 & 2^2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -3 & -1 \end{bmatrix}$, $\begin{bmatrix} 2\times 5 & 2^2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ -10 & -4 \end{bmatrix}$.

For $n \in \{-1, 0\}$, α divides β (case previously discussed), for $n = -2, -3$ we have $\begin{bmatrix} -6 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix}$ and $\begin{bmatrix} -10 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ -10 & -4 \end{bmatrix}$, respectively. □

This way we have infinitely many such examples of IN matrices in $\mathbb{M}_2(\mathbb{Z})$ which are not NI. Analogous examples can be given replacing the prime number 2 by an odd prime number. Clearly, it follows from Theorem 15 that, for any given β and large enough α (such that β divides α^2), the corresponding integral matrix $\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$ is IN but not NI.

4.2. **An application.** In this subsection we use some of our results proved in the previous sections.

We start with the matrix $A = \begin{bmatrix} 91 & 14 \\ 273 & 42 \end{bmatrix}$. Using Theorem 2 with $c = 3$, $a = 1$, $b = 0$, we get an IN-decomposition: $A = ET = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 91 & 14 \\ 13 & 2 \end{bmatrix}$.

Next we use Proposition 9, which gives $E = PE_{11}P^{-1}$ for $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$. This way $PAP^{-1} = PEP^{-1}PTP^{-1} = E_{11}\begin{bmatrix} 14 & 4 \\ -49 & -14 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 0 & 0 \end{bmatrix}$.

Finally we use our characterization Theorem 15 (actually, the special cases mentioned after its proof). The common divisors of 14, 4 are $\{\pm 1, \pm 2\}$. None of the divisibilities (i.e., $14 \mid 2\times 1$ or $14 \mid 2\times 3$ or $7 \mid 1$ or $7 \mid 3$) holds, so the initial matrix A is IN but not NI.

A **direct proof** showing the IN matrix $A = E_{11}\begin{bmatrix} 14 & 4 \\ -49 & -14 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 0 & 0 \end{bmatrix}$ has no NI-decomposition over $\mathbb{Z}$, is obtained taking $n = 3$ in Proposition 16: $(7l + 3)y = -2$ has no integer solution.

Remark. As computer shows, $\begin{bmatrix} 14 & 4 \\ 0 & 0 \end{bmatrix}$ is the "minimal" (in absolute value of the entries) example of integral IN-matrix of $\mathbb{M}_2(\mathbb{Z})$ which is not NI.

There is no conflict of interests.

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