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UNIPOTENT DIAGONALIZATION OF MATRICES

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Abstract. An element u of a ring R is called *unipotent* if $u-1$ is nilpotent. Two elements $a, b \in R$ are called *unipotent equivalent* if there exist unipotents $p, q \in R$ such that $b = q^{-1}ap$. Two square matrices A, B are called *strongly* unipotent equivalent if there are unipotent triangular matrices P, Q with $B = Q^{-1}AP$.

In this paper, over commutative reduced rings, we characterize the matrices which are strongly unipotent equivalent to diagonal matrices. For 2×2 matrices over Bézout domains, we characterize the nilpotent matrices unipotent equivalent to some multiples of E_{12} and the nontrivial idempotents unipotent equivalent to E_{11}.

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1. Introduction

Let R be an associative unital ring. For a ring R, $U(R)$ denotes the set of all the units of R, $N(R)$ the set of all nilpotents of R and E_{ij} denotes the $n \times n$ matrix with all entries zero excepting the (i, j) entry, which is 1. For a square matrix A, $\gcd(A)$ denotes the greatest common divisor of the entries of A.

In Ring Theory, two elements $a, b \in R$ are called *equivalent* if there exist two units p, q of R such that $b = q^{-1}ap$ and *conjugate* if $q = p$. An element u is called *unipotent* if $u - 1$ is nilpotent. It is well-known that unipotents are units.

In linear algebra, two rectangular $m \times n$ matrices A and B are called *equivalent* if $B = Q^{-1}AP$ for some invertible $n \times n$ matrix P and some invertible $m \times m$ matrix Q and *similar* if $m = n$ and $Q = P$. Matrix equivalence is an *equivalence relation* on the set of rectangular matrices.

Specializing the above definition, a square $n \times n$ matrix U is *unipotent* if $U = I_n + T$ for an $n \times n$ nilpotent matrix T. Unipotent matrices are invertible.

Definition. Two elements $a, b \in R$ are called *unipotent equivalent* (*u-equivalent*, for short) if there exist two unipotents p, q of R such that $b = q^{-1}ap$. Thus, a, b are u-equivalent iff there exist two nilpotents s, t of R such that $(1+t)b = a(1+s)$.

Specializing, two elements $a, b \in R$ are called *unipotent conjugate* (*u-conjugate*, for short) if there is an unipotent u such that $b = u^{-1}au$. Equivalently, $a, b \in R$ are u-conjugate iff a nilpotent $t \in R$ exists such that $b = (1+t)^{-1}a(1+t)$.

In particular, two rectangular $m \times n$ matrices A and B are called *unipotent equivalent* (*u-equivalent*, for short) if $B = Q^{-1}AP$ for some *unipotent* $n \times n$ matrix P and some *unipotent* $m \times m$ matrix Q. Specializing, two square matrices A, B are called *unipotent similar* (*u-similar*, for short) if there is a unipotent matrix U such that $B = U^{-1}AU$. Specializing again, a square matrix is called *u-diagonalizable* if it is u-similar to a diagonal matrix.

Definition. A matrix will be called *ue-diagonalizable* if it is *u-equivalent to a diagonal matrix*. Notice that u-diagonalizable matrices are ue-diagonalizable.

It is well-known that for two rectangular matrices of the same size *over a field*, their equivalence can be characterized by any of the following conditions:

The matrices can be transformed into one another by a combination of elementary row and column operations.

The matrices have the same rank.

Deciding whether two given matrices of the same size have the same rank is solved at undergraduate level by the (reduced) row echelon form.

More, an $n \times n$ matrix A has rank r iff there exist invertible matrices P, Q such that $PAQ = \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ with I_r the $r \times r$ identity matrix.

These two invertible matrices are obtained using elementary *row* operations, as for P, and using elementary *column* operations, as for Q.

As examples in section 2 show, P and/or Q may *not* be unipotent because matrices obtained by using elementary row (or column) operations on one copy of I_n (are invertible but) may *not* be unipotent.

Therefore, in general (but some exceptions will be emphasized in the sequel), the above procedure (including elementary row or column operations) does *not* work when dealing with the u-equivalence of a matrix with a diagonal one, neither when trying to characterize the u-equivalence of two matrices.

An example in the last section shows that (idempotent 2×2) matrices of the same rank may *not* be u-equivalent.

Thus, in order to obtain results concerning u-equivalent matrices we can only use the definition, that is, find two nilpotent matrices T, S and $B = (I_n+T)^{-1}A(I_n+S)$. Excepting the case of 2×2 matrices, u-equivalence for 3×3 (or higher order) matrices already amount to difficult problems.

In this paper, we mainly focus on an *analogue* of the above mentioned result on rank r matrices A, i.e. $PAQ = \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$.

We characterize some classes of ue-diagonalizable matrices, which we call *strongly ue-diagonalizable* matrices, namely, those for which P, Q are unipotent triangular matrices.

It is well-known that over Bézout domains (definition in the last section), nilpotent 2×2 matrices are similar to multiples of E_{12}, and nontrivial idempotent 2×2 matrices are similar to E_{11}. In the last section, we describe the nilpotent 2×2 matrices that are u-equivalent to a multiple of E_{12}, respectively, the nontrivial idempotent 2×2 matrices that are u-equivalent to E_{11}.

2. Examples

It is not so easy to work with unipotents because (unlike units) minus unipotents and products of unipotents may *not* be unipotent. We gather here some simple observations and examples on unipotents.

Proposition 2.1. *Inverses and powers of unipotents are unipotent in any ring.*

Proposition 2.2. *In any ring R, the following conditions are equivalent:*
(i) the opposite of every unipotent is unipotent;
(ii) the opposite of some unipotent is unipotent;
(iii) 2 is nilpotent.

As an example, in any *nil clean* ring (i.e., every element is a sum of an idempotent and a nilpotent), 2 is (central) nilpotent (see [2], Proposition 3.14).

A ring is called *NR* if $N(R)$ is a subring of R. Some examples are: commutative rings, reduced rings, (nil-)Armendariz rings, UU rings. Matrix rings are not NR.

Proposition 2.3. *In a ring R, products of unipotents are unipotent iff R is NR.*

Therefore, viewed as a binary relation, the u-equivalence is generally reflexive and symmetric, but may *not* be transitive.

Example 2.4. *Products of unipotents (with not commuting nilpotents) may* not *be unipotent.*

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = I_2 + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \text{ is not unipotent.}$$

It is easy to see that units are equivalent in any ring, and in particular, every unit is equivalent to 1. It is also easy to see that unipotents are u-equivalent to 1 in any ring.

Proposition 2.5. *The units of a ring that are u-equivalent to* 1 *are precisely the products of two unipotents.*

Recall (see [1], Corollary 1.8) that *a square matrix over a commutative ring is nilpotent iff all coefficients of the characteristic polynomial, excepting the one of t^n, are nilpotent.* It follows that over any commutative reduced ring (e.g., any integral domain), a square matrix is nilpotent iff all its eigenvalues are zero. Equivalently, its characteristic polynomial is t^n. Therefore, a square matrix M over any commutative reduced ring is a *unipotent matrix* iff its characteristic polynomial $P(t)$ is a power of $t-1$. Thus, over any commutative reduced ring, all the eigenvalues of a unipotent matrix are 1 and so $\det(M) = 1$ and $\mathrm{Tr}(M) = n$. These are necessary but not sufficient conditions.

Next, the examples mentioned in the Introduction.

Example 2.6. *Unipotence of matrices is not invariant under elementary row (or column) operations.*

Since unipotent $n \times n$ matrices over commutative reduced rings have determinant $= 1$ and trace $= n$, it is easy to give examples of elementary row (or column) operations that change the (sign of the) determinant and/or the trace. However, if we perform $(-1)\mathrm{row}_1 + \mathrm{row}_3$ on the unipotent matrix $U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, it does not change the trace nor the determinant, but $U' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$ is not unipotent, since $U' - I_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ is invertible.

Example 2.7. *Conjugation and u-equivalence are independent notions.*

1) Since $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} I_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, the idempotent $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and (not idempotent over any nonzero ring) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ are *u-equivalent but not conjugate.*

This is also an example of *u-equivalent matrices that are not u-conjugate.* Moreover, over $\mathbb{Z}_2$ this is an idempotent u-equivalent to a nilpotent.

2) Consider the nilpotent $T = \begin{bmatrix} 10 & 4 \\ -25 & -10 \end{bmatrix}$ and the 2×2 nilpotent E_{12}. Since

$$\begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix} T = \begin{bmatrix} 5 & 2 \\ 0 & 0 \end{bmatrix} = E_{12} \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix},$$

T and E_{12} are conjugate. We will show that these nilpotents are *not* u-equivalent after proving a general result, Corollary 4.5.

Example 2.8. *Conjugate units may not be u-conjugate.*

Over any commutative (unital) ring, for $V = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ we have $U' = V^{-1}UV = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$, so U and U' are conjugate. In order to check that U and U' are *not* u-conjugate, we start with

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1+x & y \\ z & 1-x \end{bmatrix} = \begin{bmatrix} 1+x & y \\ z & 1-x \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

and $x^2 + yz = 0$, which in turn reduces to $y = -2$, $z = x + 1$ and $(x - 1)^2 = 3$. Therefore, U and U' are not u-conjugate over any (commutative) ring such that 3 is not a square (e.g., $\mathbb{Z}$).

Remark. If in a ring $u^{-1}au = b$ and u is a unit but not unipotent, we still can have $v^{-1}av = b$ with unipotent v.

A trivial example is $u \in U(R) - (1 + N(R))$ since $1^{-1}u1 = u^{-1}uu$ with unipotent 1. A less trivial example is the above example over any ring such that 3 is a square (e.g. $\mathbb{Z}[\sqrt{3}]$).

Example 2.9. *Ue–diagonalizable matrices.*

Some matrices are not diagonalizable over any field, most notably the nonzero *nilpotent* matrices. For instance, consider $T = E_{12}$.

Clearly E_{12} is equivalent to E_{11}, which is in diagonal form. We just swap the columns and so $E_{11} = I_2 E_{12} U$ with $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

These matrices are also u-equivalent since $\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} E_{12} = E_{11} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ with arbitrary y. Since u-similar matrices are similar, E_{12} is not u-diagonalizable.

Example 2.10. *A ue-diagonalizable unit which is not unipotent.*

Take $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ over any commutative ring R with $2 \in U(R)$. The usual diagonalization procedure for two different eigenvalues ± 1 and linearly independent eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ gives $P^{-1}UP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ for $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $P^{-1} = 2^{-1} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Next, since

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} U = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \operatorname{diag}(1, -1) \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix},$$

it follows that U is ue-diagonalizable matrix over any commutative ring.

Example 2.11. *A diagonalizable matrix which is not u-diagonalizable.*

Take $A = \begin{bmatrix} 5 & 6 \\ -1 & 0 \end{bmatrix}$ over $\mathbb{Z}$. For eigenvalues 2, 3, eigenvectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $P = \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix}$ we have $P^{-1}AP = \operatorname{diag}(2, 3)$.

However, $\begin{bmatrix} 1+x & y \\ z & 1-x \end{bmatrix} A = \operatorname{diag}(2, 3) \begin{bmatrix} 1+x & y \\ z & 1-x \end{bmatrix}$ amounts to $x = 1 - 2z$, $y = 6(1-z)$, which replaced in $x^2 + yz = 0$ gives $2z^2 - 2z - 1 = 0$ with no integer solutions.

3. u-equivalence with triangular unipotents

We have already mentioned (see Introduction) that the procedure which consists of elementary row or column operations does *not* (in general) work when dealing with the u-equivalence of a matrix with a diagonal one. However, if a matrix A is upper triangularizable using only elementary row operations $r\mathrm{row}_i + \mathrm{row}_j$ with $i < j$ and then diagonalizable using only elementary column operations $r\mathrm{col}_i + \mathrm{col}_j$ with $i < j$ then A is *ue*-diagonalizable.

A special class of unipotent matrices consists of upper (or lower) triangular matrices. In the sequel, we consider such matrices having only 1's on the diagonal. As noticed in the previous section, this is always the case *over commutative reduced rings.*

Thus, triangular unipotents are sums $I_n + T$ where T is a strictly upper (or lower) triangular (nilpotent) matrix.

Notice that lower triangular unipotents can be obtained using only one type of elementary row operations on I_n, namely, $r\mathrm{row}_i + \mathrm{row}_j$ with $i < j$, and analogously, upper triangular unipotents using the same type of elementary column operations.

Recall that a matrix A was called *strongly ue-diagonalizable*, if there exist triangular unipotent matrices P and Q *with only* 1*'s on the diagonal* (condition which is implicitly assumed in the sequel) such that QAP is a diagonal matrix. As mentioned before, triangular unipotent matrices are of this sort over commutative reduced rings.

Since different characterizations occur, we must be more precise. A matrix is *LU* strongly ue-diagonalizable if Q is lower triangular and P is upper triangular. We define *UU*, *UL* and *LL* strongly ue-diagonalizable matrices by requiring Q, P to be (both) upper triangular, Q upper triangular and P lower triangular, or Q, P (both) lower triangular, respectively.

Since on the left and on the right, such matrices form subgroups, *these binary relations are equivalence relations.*

We first characterize the matrices that are LU strongly u-equivalent to some diagonal matrix, over commutative rings.

Theorem 3.1. *An* $n \times n$ *matrix* $A = [a_{ij}]_{1\le i,j\le n}$ *over a commutative ring is* *LU strongly ue-diagonalizable iff for every* $2 \le m \le n$, A *and* A^T *satisfy the conditions* $\begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{m-1,m} \end{bmatrix}$ *is a linear combination of* $\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m-1,1} \end{bmatrix}$, $\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m-1,2} \end{bmatrix}$, ..., $\begin{bmatrix} a_{1,m-1} \\ a_{2,m-1} \\ \vdots \\ a_{m-1,m-1} \end{bmatrix}$. *If this holds,* a_{11} *divides all the entries in the first row and in the first column.*

Proof. If $Q = [l_{ij}]$ is an $n \times n$ lower triangular unipotent matrix and $P = [r_{ij}]$ is an $n \times n$ upper triangular unipotent matrix, then QAP is diagonal iff for every

$2 \leq m \leq n$, $r_{1m}\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m-1,1} \end{bmatrix} + r_{2m}\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m-1,2} \end{bmatrix} + ... + r_{m-1,m}\begin{bmatrix} a_{1,m-1} \\ a_{2,m-1} \\ \vdots \\ a_{m-1,m-1} \end{bmatrix} +$
$\begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{m-1,m} \end{bmatrix} = \mathbf{0}$

and $l_{m1}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,m-1} \end{bmatrix} + l_{m2}\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2,m-1} \end{bmatrix} + ...$
$+ l_{m,m-1}\begin{bmatrix} a_{m-1,1} & a_{m-1,2} & \cdots & a_{m-1,m-1} \end{bmatrix} + \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{m,m-1} \end{bmatrix} =$
$\mathbf{0}$.

For $m = 2$ we get $a_{11} \mid a_{12}$ and $a_{11} \mid a_{21}$, for $m = 3$ we get $a_{11} \mid a_{13}$ and $a_{11} \mid a_{31}$ and so on. For $m = n$ we obtain $a_{11} \mid a_{1n}$ and $a_{11} \mid a_{n1}$. □

The conditions on A^T in the theorem mean $\begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{m,m-1} \end{bmatrix}$ is a linear combination of $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,m-1} \end{bmatrix}$,
$\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2,m-1} \end{bmatrix}$, ..., $\begin{bmatrix} a_{m-1,1} & a_{m-1,2} & \cdots & a_{m-1,m-1} \end{bmatrix}$.

Corollary 3.2. *A 2×2 matrix $A = [a_{ij}]_{1 \leq i,j \leq 2}$ is LU strongly ue-diagonalizable iff a_{11} divides both a_{12} and a_{21}.*

A symmetric theorem can be proved taking upper triangular unipotent matrices on the left and lower triangular unipotent matrices on the right. However, *it cannot be obtained by transpose* because $(QAP)^T = P^T A^T Q^T$, for lower triangular Q and upper triangular P, still has a lower triangular matrix on the left and an upper triangular matrix on the right. Actually, the symmetry is with respect to the *secondary* diagonal.

Theorem 3.3. *An $n \times n$ matrix $A = [a_{ij}]_{1 \leq i,j \leq n}$ over a commutative ring is UL strongly ue-diagonalizable iff for every $1 \leq m \leq n-1$, A and A^T satisfy the* *conditions* $\begin{bmatrix} a_{m+1,m} \\ a_{m+2,m} \\ \vdots \\ a_{n,m} \end{bmatrix}$ *is a linear combination of* $\begin{bmatrix} a_{m+1,m+1} \\ a_{m+2,m+1} \\ \vdots \\ a_{n,m+1} \end{bmatrix}$, $\begin{bmatrix} a_{m+1,m+2} \\ a_{m+2,m+2} \\ \vdots \\ a_{n,m+2} \end{bmatrix}$,

..., $\begin{bmatrix} a_{m+1,n} \\ a_{m+2,n} \\ \vdots \\ a_{nn} \end{bmatrix}$.

The conditions on A^T in the theorem mean $\begin{bmatrix} a_{m,m+1} & a_{m,m+2} & \cdots & a_{mn} \end{bmatrix}$ is a linear combination of $\begin{bmatrix} a_{m+1,m+1} & a_{m+1,m+2} & \cdots & a_{m+1,n} \end{bmatrix}$,
$\begin{bmatrix} a_{m+2,m+1} & a_{m+2,m+2} & \cdots & a_{m+2,n} \end{bmatrix}$, ..., $\begin{bmatrix} a_{n,m+1} & a_{n,m+2} & \cdots & a_{nn} \end{bmatrix}$.

Remarks. 1) In the $n \times n$ LU case, there is no condition on a_{nn}, and in the $n \times n$ UL case, there is no condition on a_{11}.

2) a_{11} dividing all the entries in the first row and in the first column and only one linear combination (for A or for A^T, but not for both) are necessary but not sufficient conditions for the LU case. For example, take $\begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 3 \\ 2 & 4 & 0 \end{bmatrix}$. Here $\begin{bmatrix} 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \end{bmatrix}$ but $\begin{bmatrix} 4 \\ 3 \end{bmatrix} \neq k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + l \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, over any nonzero ring.

3) As one might expect, there are matrices u-equivalent to diagonal matrices which are not LU strongly ue-diagonalizable.

For example, take $A = \begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix}$ over $\mathbb{Z}$. By the above theorem, since $2 \nmid 3$, A is not LU strongly ue-diagonalizable. However, since $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, A is u-equivalent to $\operatorname{diag}(3, -4)$. Notice that the right unipotent is lower triangular.

Example 3.4. *A LU strongly ue-diagonalizable* 3×3 *matrix.*

Take $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & -1 \\ 3 & 11 & 5 \end{bmatrix}$. Since $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 & 11 \end{bmatrix} = 4\begin{bmatrix} 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 1 \end{bmatrix}$, by the above theorem, A is LU strongly u-equivalent to a 3×3 diagonal matrix. Below we find explicitly the triangular unipotent matrices P and Q.

The sequence $-\text{row}_1 + \text{row}_2$, $-3\text{row}_1 + \text{row}_3$ and $\text{row}_2 + \text{row}_3$ gives an upper triangular matrix. Next $-3\text{col}_1 + \text{col}_2$, $-\text{col}_1 + \text{col}_3$ and $-\text{col}_2 + \text{col}_3$ gives the diagonal form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The same elementary row operations performed on I_3 give $Q = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ and on columns, $P = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. Notice that the $i < j$ condition *is essential* in order to obtain (triangular) unipotents Q and P with 1's on the diagonal.

Next, we characterize the matrices that are UU strongly ue-diagonalizable, over commutative rings.

Theorem 3.5. *An $n \times n$ matrix $A = [a_{ij}]_{1 \le i,j \le n}$ over a commutative ring is UU strongly ue-diagonalizable iff it is upper triangular, for every $i < j$, a_{ij} is a linear combination of the diagonal entries a_{ii}, $a_{i+1,i+1}$, ..., a_{jj} and if $a_{ij} = m_i^{(ij)} a_{ii} + m_{i+1}^{(ij)} a_{i+1,i+1} + ... + m_j^{(ij)} a_{jj}$ denote these linear combinations, then for every k such that $i < k < j$, $m_k^{(ij)} = m_k^{(ik)} m_k^{(kj)}$.*

Proof. We start with two upper triangular unipotent $n \times n$ matrices, Q and P, such that $QAP = D$ is diagonal. Writing $A = Q^{-1}DP^{-1}$ shows that A must be upper triangular too.

If $Q = [l_{ij}]_{1 \le i,j \le n}$ and $P = [r_{ij}]_{1 \le i,j \le n}$ are $n \times n$ upper triangular unipotent matrices, then QAP is diagonal iff the following relations hold.

$a_{k,k+1} = -r_{k,k+1} a_{kk} - l_{k,k+1} a_{k+1,k+1}$,

$a_{k,k+2} = (r_{k,k+1} r_{k+1,k+2} - r_{k,k+2}) a_{kk} + r_{k+1,k+2} l_{k,k+1} a_{k+1,k+1} + (l_{k,k+1} l_{k+1,k+2} - l_{k,k+2}) a_{k+2,k+2}$,

$a_{k,k+3} = (-r_{k,k+1} r_{k+1,k+2} r_{k+2,k+3} + r_{k,k+1} r_{k+1,k+3} + r_{k,k+2} r_{k+2,k+3} - r_{k,k+3}) a_{kk}$
$-l_{k,k+1}(r_{k+1,k+2} r_{k+2,k+3} - r_{k+1,k+3}) a_{k+1,k+1}$
$-r_{k+2,k+3}(l_{k,k+1} l_{k+1,k+2} - l_{k,k+2}) a_{k+2,k+2}$

$+(-l_{k,k+1}l_{k+1,k+2}l_{k+2,k+3}+l_{k,k+1}l_{k+1,k+3}+l_{k,k+2}l_{k+2,k+3}-l_{k,k+3})a_{k+3,k+3}$ and so on. That is, each entry over the diagonal is a linear combination of diagonal entries, as recorded above.

If these linear combinations are given, namely $a_{ij}=m_i^{(ij)}a_{ii}+m_{i+1}^{(ij)}a_{i+1,i+1}+\ldots+m_j^{(ij)}a_{jj}$, then the entries of Q and P can be expressed in terms of the m's iff $m_k^{(ij)}=m_k^{(ik)}m_k^{(kj)}$, as follows:

$r_{k,k+1}=-m_k^{(k,k+1)}$, $l_{k,k+1}=-m_{k+1}^{(k,k+1)}$,

$r_{k,k+2}=m_k^{(k,k+1)}m_{k+1}^{(k+1,k+2)}-m_k^{(k,k+2)}$, $l_{k,k+2}=m_{k+1}^{(k,k+1)}m_{k+2}^{(k+1,k+2)}-m_{k+2}^{(k,k+2)}$,

$r_{k,k+3}=-m_k^{(k,k+1)}m_{k+1}^{(k+1,k+2)}m_{k+2}^{(k+2,k+3)}+m_k^{(k,k+1)}m_{k+1}^{(k+1,k+3)}+m_k^{(k,k+2)}m_{k+2}^{(k+2,k+3)}$

$l_{k,k+3}=-m_{k+1}^{(k,k+1)}m_{k+2}^{(k+1,k+2)}m_{k+3}^{(k+2,k+3)}+m_{k+1}^{(k,k+1)}m_{k+3}^{(k+1,k+3)}+m_{k+2}^{(k,k+2)}m_{k+3}^{(k+2,k+3)}$

and so on. $\square$

Since we apply this for a 3×3 example below, we provide the formulas which give the unipotent upper triangular matrices, given the linear combinations of diagonal entries.

If $\begin{bmatrix}1&x&z\\0&1&y\\0&0&1\end{bmatrix}A\begin{bmatrix}1&s&t\\0&1&v\\0&0&1\end{bmatrix}$ is a diagonal matrix and $a_{12}=m_1^{(12)}a_{11}+m_2^{(12)}a_{22}$, $a_{23}=m_2^{(23)}a_{22}+m_3^{(23)}a_{33}$, $a_{13}=m_1^{(13)}a_{11}+m_2^{(13)}a_{22}+m_3^{(13)}a_{33}$ are the given linear combinations then $x=-m_2^{(12)}$, $y=-m_3^{(23)}$, $z=m_2^{(12)}m_3^{(23)}-m_3^{(13)}$, $s=-m_1^{(12)}$, $v=-m_2^{(23)}$, $t=m_1^{(12)}m_2^{(23)}-m_1^{(13)}$.

The LL characterization (and proof), symmetric with respect to the secondary diagonal, is the following

Theorem 3.6. *An $n\times n$ matrix $A=[a_{ij}]_{1\leq i,j\leq n}$ over a commutative ring is LL strongly ue-diagonalizable iff it is lower triangular, for every $i>j$, a_{ij} is a linear combination of the diagonal entries a_{ii}, $a_{i+1,i+1}$, ..., a_{jj} and if $a_{ij}=m_i^{(ij)}a_{ii}+m_{i+1}^{(ij)}a_{i+1,i+1}+\ldots+m_j^{(ij)}a_{jj}$ denote these linear combinations, then for every k such that $i>k>j$, $m_k^{(ij)}=m_k^{(ik)}m_k^{(kj)}$.*

Remark. Having characterized, in the four possible cases, all matrices that are strongly ue-diagonalizable, it is easy to provide 3×3 integral examples which satisfy one condition but *not* the other three.

Example 3.7. *A 3×3 matrix which is UU but not LU nor UL nor LL strongly ue-diagonalizable.*

For instance, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ over $\mathbb{Z}$, is *not* LU strongly ue-diagonalizable since $\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{4}{3}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, is the unique linear combination, but is UU strongly u-equivalent to

$$\mathrm{diag}(1,3,5) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Here the formulas previously displayed are used in order to find the unipotent upper triangular left and right matrices.

It is obviously *not* LL strongly ue-diagonalizable since it is not lower triangular, and *not* UL ue-diagonalizable. Indeed, notice that the conditions in the 3×3 case require $\begin{bmatrix} a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to be a linear combination of $\begin{bmatrix} a_{22} \\ a_{32} \end{bmatrix}$, $\begin{bmatrix} a_{23} \\ a_{33} \end{bmatrix}$ which trivially holds and $\begin{bmatrix} a_{12} & a_{13} \end{bmatrix}$ to be a linear combination of $\begin{bmatrix} a_{22} & a_{23} \end{bmatrix}$, $\begin{bmatrix} a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & a_{33} \end{bmatrix}$. For the latter, $a_{33} \mid a_{23}$ is necessary, but $5 \nmid 4$.

According to the above characterizations, Example 2.10, is a 2×2 matrix which is ue-diagonalizable but is not strongly ue-diagonalizable (that is, not LU, not UL, not UU and nor LL).

It is harder to find an

Example 3.8. *of* 3×3 *matrix which is ue-diagonalizable but not strongly ue-diagonalizable*

and quasi-impossible without computer aid.

Especially, 3×3 matrices with zero NE entry cannot be delt with elementary row (or column) operations with $i < j$.

1) The matrix $A = \begin{bmatrix} 0 & -1 & 1 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is ue-diagonalizable since

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{diag}(1,2,1).$$

It is obviously not UU or LL strongly ue-diagonalizable (nor being triangular), and not LU since the NE entry ($= 0$) does not divide all entries in the first row (or column). Finally, it is not UL since $\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not an integral linear combination. Thus, A is not strongly ue-diagonalizable over $\mathbb{Z}$.

2) For $U = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, computer aid was necessary.

Since $\begin{bmatrix} 0 & -2 & -1 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} U \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} = \mathrm{diag}(-1, 1, 1)$, U is ue-diagonalizable. As in the previous example, it can be shown that U is not strongly ue-diagonalizable.

If for both unipotents we permit only entries $-2, -1, 0, 1, 2$ then the computer produced 232 possible ue-diagonalizations for U, none if we restrict the entries only to $-1, 0, 1$.

4. u-equivalence for 2×2 nilpotent or idempotent matrices

Important Ring Theory properties, for instance being idempotent, being nilpotent or being a unit, are known to be invariant for conjugation but not for equivalence. These are also not invariant to u-equivalence, as shown by easy examples: $E_{11} = E_{12}\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ for idempotent-nilpotent,

The u-equivalence of the units in any ring was addressed in Section 2.

Generalizing some of our previous examples, we describe the 2×2 nilpotent matrices that are u-equivalent to multiples of E_{12} and the 2×2 nontrivial idempotent matrices that are u-equivalent to E_{11}.

Definition. A ring R is called a *GCD* ring if every pair of elements has a greater common divisor, and a GCD ring is called a *Bézout* ring if the greater common divisor of any two elements is a linear combination of these.

An integral domain R is called *UFD* (unique factorization domain) if every nonzero non-unit element can be written as a product of prime elements (or irreducible elements), uniquely up to order and units. Recall that both Bézout domains and UFD's are GCD domains.

We first list some useful properties of elements a, b, c in a GCD *domain* such that $a^2 + bc = 0$. As customarily, equality denotes association (in divisibility), that is $a = b$ means $a = bu$ for some unit $u \in U(R)$.

Lemma 4.1. *Let R be a GCD domain and $a, b, c \in R$ such that $a^2 + bc = 0$ with $a \neq 0$. Then*

(i) $\gcd(b, c) = \gcd(a, b, c)$.

(ii) If $0 \neq b \mid a$ *then* $b = \gcd(a, b, c)$.

(iii) If R *is a UFD and* $\gcd(a, b, c) = 1$ *then* ($\gcd(b, c) = 1$ *and) regardless of sign,* b, c *are squares.*

Proof. (i) Let $d = \gcd(b, c)$ and write $b = db_1$, $c = dc_1$. Then $a^2 = -b_1c_1d^2$ so that $d^2 \mid a^2$. Hence $d \mid a$.

(ii) Write $a = br$. Then $0 = a^2 + bc = b^2r^2 + bc$ which by cancellation gives $c = -br^2$. Hence $b \mid c$.

(iii) By (i), $\gcd(b, c) = 1$ and since $-bc$ is a square, so is each of b and c. □

Corollary 4.2. *Over any UFD, every nonzero nilpotent* 2×2 *matrix with (collectively) coprime entries is of form* $\begin{bmatrix} pq & p^2 \\ -q^2 & -pq \end{bmatrix}$ *with* $\gcd(p, q) = 1$.

Theorem 4.3. *Over any Bézout domain, a* 2×2 *(nonzero) nilpotent matrix* $T = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ *(with* $a^2 + bc = 0$*) is u-equivalent to* dE_{12} *with* $d = \gcd(T)$ *iff there exist* $x, y, z \in R$ *such that* $bx^2 + a(1 - x)y = 0$ *and* $bz = a(1 - x)$. *The solution* $(x, y) = (0, 0)$ *is suitable iff* $b \mid a$.

Proof. First observe that if $T = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ with $a^2+bc = 0$, has a zero entry, then it has (at least) 3 zero entries, so it is a multiple of E_{12} or E_{21}. We can discard this case since $(I_2 + \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix})dE_{21} = dE_{12}(I_2 + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix})$ and continue assuming T has only *nonzero* entries.

Secondly, notice that if $d = \gcd(T)$, by writing $a = da_1$, $b = db_1$, $c = dc_1$, in the equality
$\begin{bmatrix} 1+x & y \\ z & 1-x \end{bmatrix} \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = dE_{12} \begin{bmatrix} 1+s & t \\ u & 1-s \end{bmatrix} = d \begin{bmatrix} u & 1-s \\ 0 & 0 \end{bmatrix}$, we can cancel d, that is, we can suppose $\gcd(T) = 1$ (and suppress the lower indexes).

In what follows we are looking for two unipotent matrices (i.e., $x, y, z \in R$ and $s, t, u \in R$, respectively) such that
$\begin{bmatrix} 1+x & y \\ z & 1-x \end{bmatrix} \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = E_{12} \begin{bmatrix} 1+s & t \\ u & 1-s \end{bmatrix} = \begin{bmatrix} u & 1-s \\ 0 & 0 \end{bmatrix}$ with $x^2 + yz = s^2 + ut = a^2 + bc = 0$. This equality amounts to $u = (1 + x)a + yc$, $s = 1 + ya - (1 + x)b$ and $za + (1 - x)c = 0 = zb - (1 - x)a$.

Notice that since $a \neq 0 \neq c$, in any domain the last two equalities are equivalent: if we multiply $za + (1 - x)c = 0$ by a, replace $a^2 = -bc$ and cancel c, we get the second equality $0 = zb - (1 - x)a$. Analogous, conversely. Therefore in the sequel we preserve just the equality $zb = (1 - x)a$.

Next, multiplying by b in $x^2 + yz = 0$ and replacing $bz = a(1 - x)$ we get

$$bx^2 + a(1 - x)y = 0,$$

which is a binary quadratic equation for every given a, b. Clearly, this equation has at least the solution $(0, 0)$. However, this verifies $bz = a(1 - x)$ iff (not only $b \mid a^2$, which is equivalent to the matrix T being nilpotent but) $b \mid a$. According to the previous lemma, since $\gcd(T) = 1$, this holds only iff $b = 1$, i.e., $T = \begin{bmatrix} a & 1 \\ -a^2 & -a \end{bmatrix}$, which is easy to handle (more general, see remark 2, after this proof). □

Remarks. 1) Using the general theory of solving such binary quadratic equations (see [3], $D = a^2$), we perform the substitutions $a^2x = X + a^2$, $a^2y = Y + 2ab$ and obtain $X(bX - aY) = -a^4b$. Over any UFD, this gives finitely many solutions on factorizing the RHS.

Coming back to the previous corollary, we can assume ($\gcd(T) = 1$, so $\gcd(b, c) = 1$ and so) $b = p^2$, $c = -q^2$ and $a = pq$ for some $p, q \in R$ with $\gcd(p, q) = 1$. Thus the equations become $px^2 + q(1 - x)y = 0$ respectively $X(pX - qY) = -p^5q^4$ with $\gcd(p, q) = 1$. Moreover, $pz = q(1 - x)$ is necessary.

2) In the $b \mid a$ case, if $a = bd$ then $z = d$, $c = -az = -bd^2$ and (indeed) $\begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} bd & b \\ -bd^2 & -bd \end{bmatrix} = \begin{bmatrix} bd & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} u & 1-s \\ 0 & 0 \end{bmatrix}$. Observe that both matrices have $\gcd = b$.

We can provide many *integral* examples using the following

Proposition 4.4. *If q is prime and $0 < p + 1 < q$ then the quadratic equation $px^2 + q(1 - x)y = 0$ has only the solution $(0, 0)$.*

Proof. If $y = 0$ then $x = 0$, so suppose $y \neq 0$. Since p, q are coprime, $q \mid x^2$ and since q is prime, $q \mid x$. Write $x = kq$. Then $q(1 - qk) \mid pq^2k^2$ and so $qk - 1 \mid pqk^2$. Since $pqk^2 = (qk - 1)pk + pk$ it follows $qk - 1 \mid pk$.

If $k > 0$, since $p + 1 < q$, it follows $pk < qk - 1$ and so $k = 0$, a contradiction. If $k < 0$ then $qk < (p + 1)k \leq pk + 1$. Hence $qk - 1 < pk < 0$ and $k = 0$, a contradiction. □

Corollary 4.5. *Let q be prime and $2 < p+1 < q$. The (nonzero) nilpotent integral matrix $\begin{bmatrix} pq & p^2 \\ -q^2 & -pq \end{bmatrix}$ is not u-equivalent to any multiple of E_{12}.*

Coming back to Example 7 in Section 2, with the notations of the first remark above and $p = 2$, $q = 5$, the integral nilpotent $T = \begin{bmatrix} 10 & 4 \\ -25 & -10 \end{bmatrix}$ is *not* u-equivalent to the 2×2 nilpotent E_{12}. That the associated quadratic Diophantine equation $2x^2 - 5xy + 5y = 0$ has only the solution $(0,0)$, also follows using [4].

Next, in a similar way, we describe the nontrivial 2×2 idempotent matrices, which are known to be similar to E_{11}, over any ID ring (e.g., Bézout domain).

A ring R is an *ID* ring if every idempotent matrix over R is similar to a diagonal one. Examples of ID rings include: division rings, local rings, projective-free rings, principal ideal domains, elementary divisor rings, unit-regular rings and serial rings.

Theorem 4.6. *Over any Bézout domain R, a nontrivial idempotent matrix $E = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ (with $a(1-a) = bc$) is u-equivalent to E_{11} iff there exist $x, y, z \in R$ such that $ax^2 + c(x-1)y = 0$ and $za + (1-x)c = 0$. The solution $(x, y) = (0, 0)$ is suitable iff $a \mid c$.*

Proof. We start with

$$\begin{bmatrix} 1+x & y \\ z & 1-x \end{bmatrix} \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} = E_{11} \begin{bmatrix} 1+s & t \\ u & 1-s \end{bmatrix} = \begin{bmatrix} 1+s & t \\ 0 & 0 \end{bmatrix}$$

where $a(1-a) = bc$ and $x^2+yz = s^2+ut = 0$. This amounts to $(1+x)a+yc = 1+s$, $(1+x)b + y(1-a) = t$ and $za + (1-x)c = 0 = zb + (1-x)(1-a)$. The last two equalities are equivalent if $b, c \neq 0$ and $a \notin \{0, 1\}$ (multiply the first by $1-a$, replace $a(1-a) = bc$ and cancel by c, one way and so on).

So $s = (1+x)a + yc - 1$, $t = (1+x)b + y(1-a)$ and we record $za + (1-x)c = 0$ which we multiply by y, replace $yz = -x^2$ and obtain

$$ax^2 + c(x-1)y = 0.$$

Replacing $(0, 0)$ in $za + (1-x)c = 0$ gives $c = -za$, which holds iff a divides c. $\square$

As for the nilpotent matrices above, we can provide many *integral* examples using the following

Proposition 4.7. *If c is prime and $2 < a+1 < c$ then the equation $ax^2+c(x-1)y = 0$ has only the solution $(0, 0)$.*

Corollary 4.8. *If c is prime and $2 < a+1 < c$ and $a \nmid c$, the nontrivial idempotent integral matrix $E = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is not u-equivalent to E_{11}.*

The above result does *not* exhaust such (nontrivial) idempotent matrices.

Example 4.9. *A 2×2 (nontrivial) idempotent integral matrix which is not u-equivalent to E_{11}.*

Take $E = \begin{bmatrix} 6 & 2 \\ -15 & -5 \end{bmatrix}$ (i.e., $a = 6$, $c = -15$). The equation $6x^2 - 15xy + 15y = 0$ has only the solution $(0,0)$ which is not suitable since $6 \nmid 15$. Thus, E and E_{11} are not u-equivalent. Since $UE = E_{11}U$ for $U = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$, E and E_{11} are conjugate.

In closing, notice that a 2×2 idempotent can be u-equivalent to a 2×2 nilpotent (see Example 2.9).

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