Abelian groups with dual endomorphism ring

Grigore Călugăreanu [*]

November 4, 2009

Abstract

Abelian groups with dual endomorphism ring are characterized via semiperfect and finitely left dual endomorphism rings.

1 Introduction

A ring R is a *left dual* ring if every left ideal is a left annihilator (i.e., using the left and right annihilators, for every left ideal L, $l(r(L)) = L$). Right dual rings are defined similarly and a ring is *dual* if it is right and left dual (this notion comes back to Kaplansky [3]).

A ring is *right P-injective* if every left principal ideal is a left annihilator (the natural definition reads as: every map $aR \longrightarrow R_R$ extends to R). Observe that, if a left principal ideal, say Ra, is a left annihilator, then $Ra = l(r(a))$.

Just to have a term at hand, we introduce the following definition: R is *finitely left dual* if every finitely generated left ideal is a left annihilator.

A ring is *Baer* if every left (or right) annihilator is a direct summand in $_RR$ (respectively R_R) and *semisimple* if every left (or right) ideal is a direct summand (i.e., generated by an idempotent) in $_RR$ (respectively R_R). Notice that a ring is semisimple if and only if it is Baer and dual.

Finally a ring is *semiperfect* if it is semilocal and idempotents lift modulo the Jacobson radical. Notice that (by [1], Theorem 3.9), dual rings are semiperfect.

In this short note, we characterize the Abelian groups having dual endomorphism ring - via semiperfect and finitely left dual endomorphism rings .

From now on, the term "group" will designate an "Abelian group".

2 The characterization

In the sequel we shall use the following chart

$$\begin{array}{ccccc} \text{left dual} & \Longrightarrow & \text{finitely left dual} & \Longrightarrow & \text{right P-injective} \\ \Uparrow & & & & \\ \text{dual} & \Longrightarrow & \text{semiperfect} & \Longrightarrow & \text{semilocal} \end{array}$$

[*]2000 Mathematics Subject Classification: 16 S 50, 20 K 30 / Key words and Phrases: dual rings, semiperfect rings, finitely left dual rings, Abelian groups, splitting

First recall the following

(23.8) Theorem ([4]) *Let M be a right module over a ring k. Then $R = \mathrm{End}(M_k)$ is semiperfect iff M is a finite direct sum of strongly indecomposable k- modules.*

Here a module N_k is called *strongly indecomposable* if $\mathrm{End}(N_k)$ is local.

Therefore

Proposition 1 *A mixed group has semiperfect endomorphism ring if and only if it is a direct sum of a torsion and a torsion-free group (i.e., splitting), both with semiperfect endomorphism ring.*

A torsion group has semiperfect endomorphism ring exactly if it is of finite rank (finite direct sum of cocyclic groups).

A divisible group has semiperfect endomorphism ring exactly if it is of finite rank (i.e. a finite direct sum of quasicyclic groups $\mathbf{Z}(p^\infty)$ and $\mathbf{Q}$).

A torsion-free group G has semiperfect endomorphism ring if and only if it decomposes into a finite direct sum of indecomposable groups, all having local endomorphism rings.

Proof. Every strongly indecomposable Abelian group is indecomposable (see [5]) and the classical Kulikov theorems imply that there are no mixed indecomposable Abelian groups. Moreover, since any nontorsion-free group is (strongly) indecomposable if and only if it is cocyclic, it only remains to use the above mentioned Theorem. ■

Further on, recall

Theorem 2.2 [2]*The endomorphism ring of an abelian group G is finitely left dual iff $G = D \oplus R$ with torsion-free divisible D and reduced R such that:*

(i) the nonzero p-components of R are homogeneous p-groups;

(ii) $R/T(R)$ is divisible (and so R is a pure subgroup of $\prod\limits_p R_p$ which includes $\bigoplus\limits_p R_p$);

(iii) if H is any endomorphic image of R, then every (group) homomorphism $H \to R$ extends to an endomorphism of R;

(iv) if $D \neq 0$, every subgroup H of R such that R/H embeds in a finite direct sum of copies of G, has a (suitable) subgroup, generated by the image of a homomorphism $G \to H$;

(v) if a subgroup H of R is an intersection of finitely many kernels of endomorphisms of R and K is a subgroup of H which is the image of a homomorphism $R \to H$, then every subgroup M of G which is a kernel of an endomorphism of G, such that $K \subseteq M$, includes H.

Finally, here is the structure of the groups having dual endomorphism rings:

Theorem 2 *A mixed group has dual endomorphism ring if and only if it is a direct sum of a torsion and a torsion-free group (i.e., splitting), both with dual endomorphism ring.*

A torsion group has dual endomorphism ring exactly if it is of finite rank with homogeneous primary components (i.e., finite direct sums of isomorphic p-groups).

A divisible group has dual endomorphism ring exactly if it is of finite rank (i.e., a finite direct sum of quasicyclic groups $\mathbf{Z}(p^{\infty})$ *and* $\mathbf{Q}$*).*

A torsion-free group G has dual endomorphism ring if and only if it finite rank divisible (i.e., a finite direct sum of $\mathbf{Q}$*).*

Proof. Since dual rings are semiperfect, and groups with semiperfect endomorphism rings are splitting using the first two conditions (i) and (ii) we deduce right away the above statements (by routine Abelian Group Theory arguments). ∎

Among these, the groups with semisimple endomorphism rings are well-known: *all finite direct sums of finite elementary (torsion) groups and finite rank torsion-free divisible groups.*

Open problem. Find the Abelian groups having left-dual endomorphism rings.

The difficulty here is lack of information about the non-finitely generated (one-sided) ideals of $\mathrm{End}(G)$, the endomorphism ring of a group G.

References

[1] Hajarnavis C. R., Norton N. C. *On dual rings and their modules.* J. Algebra 93 (1985) no. 2, 253–266.

[2] Ivanov A. V. *Abelian groups with self-injective rings of endomorphisms and with rings of endomorphisms with the annihilator condition.* (Russian) Abelian groups and modules, 93–109, 247–248, Tomsk. Gos. Univ., Tomsk, 1981.

[3] Kaplansky I. *Dual rings.* Ann. of Math. (2) 49, (1948) 689–701.

[4] Lam T. Y. *A First Course in Noncommutative Rings.* Second edition. Graduate Texts in Mathematics, 131. Springer-Verlag, New York, 2001.

[5] Orsatti A. *Alcuni gruppi abeliani il cui anello degli endomorfismi è locale.* (Italian) Rend. Sem. Mat. Univ. Padova 35 (1965) 107–115.

Grigore Călugăreanu
Facultatea de Matematică-Informatică

Universitatea Babeş-Bolyai,
str. Kogălniceanu 1, Cluj-Napoca, ROMANIA

e-mail: calu@math.ubbcluj.ro, sau, calug2001@yahoo.com