UNITS OF A SUBRING WITH DIFFERENT IDENTITY "PRODUCE" UNITS OF THE WHOLE RING

Exercise 1. *Let S be a subring of a ring R and let 1_S be the identity of S and 1_R the identity of R.*

(1) Show that each unit of S "produces" a unit of R.

(2) If $1_S \neq 1_R$ then 1_S is a zero divisor of R.

Solution 2. *(1) Suppose $u, v \in S$ and $uv = vs = 1_S$ (i.e., $u, v \in U(S)$). Then $u - (1_R - 1_S)$ is a unit of R with inverse $v - (1_R - 1_S)$. Notice that (obviously by definition of 1_R) $1_R \cdot 1_S = 1_S \cdot 1_R = 1_S$ and so (by simple computation) $[u - (1_R - 1_S)] \cdot [v - (1_R - 1_S)] = [v - (1_R - 1_S)] \cdot [u - (1_R - 1_S)] = 1_R$.*

(2) Just observe $(1_R - 1_S) \cdot 1_S = 0$.

Remarks. 1) This correspondence defines an *injective* function from $U(S)$ to $U(R)$. Hence for the cardinals, $|U(S)| \leq |U(R)|$.

2) If e is an idempotent in a subring S, it is also an idempotent in the whole ring. Notice (again by simple computation) that also $e + 1_R - 1_S$ is an (possible different) idempotent in R.

Examples. 1) For $S = 2\mathbb{Z}_{10}$ and $R = \mathbb{Z}_{10}$, $1_S = \overline{6}$ and $1_R = \overline{1}$.

2) If R, S are rings, then the subring $R \times \{0\} \subset R \times S$ has the identity $(1_R, 0)$.