

Diagonal expansion for 3×3 matrices

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Exercise [high school]. Find a necessary and sufficient condition for a 3×3 matrix A whose determinant can be computed using the three diagonal minors, multiplied by the complement diagonal entries, that is, if $A = [a_{ij}]$, $1 \leq i, j \leq 3$ then

$$\det A = a_{11}A_{23}^{23} - a_{22}A_{13}^{13} + a_{33}A_{12}^{12}$$

where A_{ij}^{st} denotes the 2×2 minor on the rows i, j and columns s, t .

For example, $A_{23}^{23} = \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$ is one of the diagonal minors.

Solution 1 *Just computation. The necessary and sufficient condition is*

$$a_{13}A_{23}^{12} + a_{31}A_{12}^{23} = 0.$$

Examples. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & m \end{bmatrix}$ for any m ($\det = -m+7$), $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ ($\det =$
1).