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Dependent rings

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Abstract
A ring was called *right dependent* if the elements of the regular module R_R are right dependent, in the usual sense of Linear Algebra. Examples include: the commutative rings, the unit-regular rings, the rings satisfying the strong rank condition, matrix rings over commutative rings and many others. In this note, the right (resp. left) dependent rings are studied.

Keywords Dependent ring · Absolutely dependent element · IBN ring · (Strong) Rank condition · Ore domain

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1 Introduction

As early as 1968, G. Ehrlich (see [6], Theorem 6) pointed out that a necessary condition for a ring to be unit-regular is to be dependent, in the usual sense of Linear Algebra. To be more precise, we start with the following

Definition Two elements a, b in a ring R are called *right dependent*, if there are elements $s, t \in R$ not both zero such that $as + bt = 0$. A ring is called *right dependent* if every two elements are right dependent. Inversely, two nonzero elements a, b will be called *right independent* if $ax + by = 0$ holds only for $x = y = 0$. *Left (in)dependence* is defined symmetrically. We say that a ring is *(in)dependent* if it is both right and left (in)dependent.

Equivalently, these definitions can be given using the (regular) modules R_R or ${}_RR$: a, $b \in R$ are right dependent, iff these are R -dependent in R_R and left dependent, if these are R-dependent in ${}_RR$.

As far as we were able to find, no paper since [6] has studied this condition.

The classes of right (or left) dependent rings are quite large.

Obviously, *any two elements which commute are dependent* ($ab + b(-a) = 0$). Hence *commutative rings are dependent*.

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Moreover, *matrix rings over commutative rings turn out to be dependent*.

However, *the endomorphism ring of an infinite dimensional vector space over a division ring is not dependent*.

In this note we study right (resp. left) dependent rings.

Since these conditions will appear subsequently, we recall here the following

Definitions A ring R is said to have *(right) IBN* ("Invariant Basis Number") if, for any natural numbers n, m, $R^n \cong R^m$ implies $n = m$. This means that any two bases on a finitely generated free module F_R have the same (finite) number of elements (called the *rank* of F). As it turns out, the IBN condition is right-left symmetric.

A ring R satisfies *the (right) rank condition* (*RC* for short) if, for any natural number n, any set of R-module generators for $(R^n)_R$ has cardinality $\geq n$. Equivalently, if there is an epimorphism $R^n \to R^m$ in Mod-R, then $n \geq m$. This condition is also right-left symmetric.

We say that R satisfies the *right strong rank condition* (*SRC*, for short) if, for every natural number n, any set of linearly independent elements in $(R^n)_R$ has cardinality $\leq n$. Equivalently, for any n, any set of more than n vectors must be (linearly) dependent in $(R^n)_R$. Also equivalently, if there is a monomorphism of right free modules $R^m \to R^n$ then $m \leq n$. The *left strong condition* is defined symmetrically but examples show that the SRC condition is not right-left symmetric.

Finally, consider S a multiplicative(ly closed) (sub)set of R, i.e., $S \cdot S \subseteq S$, $1 \in S$ and $0 \notin S$. Then S is called *right permutable* if for any $a \in R$ and $s \in S$, $aS \cap sR \neq \varnothing$. Let S be the multiplicative set of all the non left nor right 0-divisors of R. Then R is called *right Ore* ring if S is right permutable. Symmetrically, one defines *left permutable* subsets of rings and *left Ore* rings and examples show that the Ore condition is not right-left symmetric.

Besides its properties (the usual Ring Theory constructions are discussed in Sect. 2), the class of right dependent rings (properly) includes not only the unit-regular rings but also includes the class of rings which have the right SRC and is included in the class of rings which have IBN_1, that is, rings which have a rank and this rank is 1. Refinements of the SRC, RC and IBN conditions are introduced and studied, with special emphasis to the (right) dependence condition, in Sect. 3. Several delimit examples are given in Sect. 4. To stimulate future work, some open questions are raised all over the paper.

Our main results are

A direct product ring $R \times S$ is right (or left) dependent iff so is one of R and S.

Matrix rings over right (or left) dependent rings are right (resp. left) dependent.

If $R[x]$ or $R[[x]]$ is right (or left) dependent then R is right (resp. left) dependent. The converses fail.

A domain is right (or left) dependent iff it is right (resp. left) Ore.

The right (or left) dependence property is

- not right-left symmetric
- logically independent from von Neumann regularity.

Subrings and quotient rings of right (or left) dependent rings, may not be right (resp. left) dependent.

The rings we consider are nonzero, associative and with identity. As customary $\mathbb{M}_n(R)$ denotes the matrix ring over R, and $\mathbb{T}_n(R)$ the ring of (upper) triangular matrices over R. A ring R is called *Dedekind finite* (DF for short) if for any $a, b \in R$, $ba = 1$ whenever $ab = 1$.

To simplify the writing, in the sequel we refer to right dependent elements and right dependent rings. Unless otherwise stated, the corresponding results "on the left" also hold. However, for convenience (with few exceptions), we shall continue to write "dependent" to refer to right dependent.

2 Dependence results

The binary relation of dependence on a ring R, is clearly *symmetric* and *reflexive* (on the right: $a.r + a.(-r) = 0$ for any $r \neq 0$) but trivially not transitive.

Indeed, let a, b be independent in a ring R. Then a and 0 (or 1) are dependent and 0 (resp. 1) and b are dependent.

We continue with a useful

Definition An element $a \in R$ will be called right (or left) *absolutely* dependent if for every $r \in R$, a and r are right (resp. left) dependent.

Clearly, *a ring is right (or left) dependent iff it has only right (resp. left) absolutely dependent elements*.

Clearly, *the center* $Z(R)$ of any ring R *consists only of (right or left) absolutely dependent elements*. In particular, 0 and 1 are absolutely dependent in any ring.

Moreover, if a is (right or left) absolutely dependent, so is $ac = ca$ for any $c \in Z(R)$ (indeed, $ax + ry = 0$ implies $(ac)x + r(cy) = 0$ if $x \neq 0$ and $(ac)0 + ry = 0$ if $x = 0$ and so $y \neq 0$).

A similar result holds replacing central elements with units.

Lemma 1 *If a is absolutely dependent and u is a unit then au and ua are also absolutely dependent.*

Proof Let $0 \neq r \in R$. There are $s, t \in R$ not both zero such that $(au)u^{-1}s+rt = as+rt = 0$. Since $u^{-1}s = 0$ iff $s = 0$, au and r are dependent.

As for ua, by contradiction, suppose ua and $c \in R$ are independent. Then for any $s, t \in R$, $uas + ct = 0$ implies $s = t = 0$. Since $uas + ct = 0$ is equivalent to $as + u^{-1}ct = 0$, it follows that a and $u^{-1}c$ are independent, a contradiction. □

Corollary 2 *If $b \in R$ is equivalent to a and a is absolutely dependent, so is b.*

Proof Just recall that b is equivalent to a if $b = uav$ for some units u and v. □

It turns out that there are plenty of absolutely dependent elements in any ring. Indeed

Lemma 3 *(i) Any left zero divisor is (right) absolutely dependent.*
(ii) Any right invertible element is (right) absolutely dependent.

Proof (i) If there exists $c \neq 0$ with $ac = 0$ then $ac + r.0 = 0$ for any $r \in R$.
(ii) If $ab = 1$ then $a(br) + r(-1) = 0$ for any $r \in R$. □

Since in any nonzero right Artinian ring, every element is a left zero divisor or is right invertible, we infer that *right Artinian rings are right dependent*. More general examples will follow.

Note that all the (main) special elements Ring Theory uses, are absolutely dependent: *idempotents* and *nilpotents* in any ring are (left and right) zero divisors, and *units* are right (and left) invertible.

Hence, *Boolean rings, division rings and local rings are dependent*.

The following result will be useful.

Lemma 4 *In a direct product ring $R \times S$, (a, b) is absolutely dependent iff so is one of a and b.*

Proof Suppose (say) a is absolutely dependent in R. For every $r \in R$ there are x, y not both zero such that $ax + ry = 0$. Hence for every $b, s \in S$, $(a, b)(x, 0) + (r, s)(y, 0) = (0, 0)$ and $(x, 0)$, $(y, 0)$ are not both zero. Notice that this holds for every $b \in S$, that is, if a is absolutely dependent in R, for any $b \in S$, the pair (a, b) is absolutely dependent in $R \times S$. Conversely, assume neither a nor b are absolutely dependent. There are $c \in R$, $d \in S$ with a, c independent in R and b, d independent in S. By contradiction assume (a, b) is absolutely dependent. Then there exist pairs $(x, y), (z, w) \in R \times S$, not both $(0, 0)$ such that $(a, b)(x, y) + (c, d)(z, w) = (0, 0)$. However, by independence, $ax + cz = 0$ implies $x = z = 0$ and $by + dw = 0$ implies $y = w = 0$, that is both $(x, y) = (z, w) = (0, 0)$, a contradiction. □

Using this lemma we can prove a result for direct products of dependent rings, perfectly analogous to the one proved for rings with the strong rank condition (see **1.33** [9]): a direct product $R = A \times B$ satisfies the (right) SRC iff one of A, B does.

Proposition 5 *A direct product ring $R \times S$ is dependent iff so is one of R and S.*

Proof If (say) R is dependent and $(a, b) \in R \times S$, then a is absolutely dependent and so is (a, b) by the previous lemma. Conversely, if neither R nor S are dependent, so is $R \times S$, again by the previous lemma. □

This extends to arbitrary direct products of dependent rings and, as in the SRC special case (see **1.34** [9]), implies

Corollary 6 *If R is a dependent ring, so is $R \times S$ for any ring S.*

Question Characterize the *indecomposable* dependent rings, that is, those which have no nontrivial central idempotents.

Remark We noticed that products of absolutely dependent elements with central elements or with units, are also absolutely dependent. However, *a product of two absolutely dependent elements, may not be absolutely dependent.*

Indeed, according to the previous lemma, if we take not absolutely dependent $a \in R$ and $b \in S$, then $(a, 1)$ and $(1, b)$ are absolutely dependent in $R \times S$ because 1 is, but their product, (a, b), is not.

Some other cases of absolutely dependent products are recorded in the following

Lemma 7 *(1) If a is absolutely dependent and b is right invertible then ab is absolutely dependent. Equivalently, if ab and c are independent and b is right invertible then a and c are independent.*

(2) If b is a left zero divisor then for every $a \in R$, ab is absolutely dependent.

(3) If a is absolutely dependent and b is a left zero divisor then ab is absolutely dependent.

Proof (1) By contradiction, suppose ab and c are independent. Then for every $s, t \in R$, $(ab)s + ct = 0$ implies $s = t = 0$. Since $bR = R$, this implies that a and c are independent.

(2) Obviously ab is also a left zero divisor and we use the previous lemma.

(3) Suppose $bc = 0$ for $c \neq 0$. Then $(ab)c + r.0 = 0$ for any $r \in R$, shows that ab is absolutely dependent. □

We can adapt the proof from [6] for

Proposition 8 *In any ring, every unit-regular element is absolutely dependent.*

Proof Let a be unit-regular. If a has a right inverse, it is absolutely dependent by previous lemma. If a has no right inverse, write $a = eu$ for $e = e^2$ and unit u. Then $au^{-1}(1-e) = 0 = au^{-1}(1-e) + r.0 = 0$ for any $r \in R$. Here $u^{-1}(1-e) \neq 0$. Otherwise $u^{-1}(1-e) = 0$ implies $e = 1$ and so $a = u$, a contradiction.

The left case is symmetric. □

Corollary 9 *([6]) Unit-regular rings are dependent.*

Remarks (1) It was noted in [6] that $\mathbb{Z}(n)$ is (unit-)regular iff n is square-free. Hence $\mathbb{Z}(n)$ with any not square-free n is *dependent but not unit-regular*.

(2) Since the following inclusions of classes of rings are well-known {one sided Artinian} $\subseteq$ {semiprimary} $\subseteq$ {right or left perfect} $\subseteq$ {strongly regular} $\subseteq$ {strongly π-regular} $\subseteq$ {unit-regular}, all these classes of rings are dependent. In particular, *finite rings are dependent*.

(3) The following result may be found in [6]: *If a has no right inverse and is unit-regular then a is a left zero divisor.*

This result cannot be generalized, replacing "unit-regular" with "right dependent". Otherwise, the (right) absolutely dependent elements would be precisely those in Lemma 3: the right invertible elements and the left zero divisors. This fails for instance in $\mathbb{M}_2(\mathbb{Z})$.

According to Corollary 12 (below, after the next theorem), $\mathbb{M}_2(\mathbb{Z})$ is dependent, that is, all integral 2×2 matrices are absolutely dependent. However $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ is neither right invertible nor left zero divisor.

Questions (1) Describe the rings all whose elements in the Jacobson radical are absolutely dependent.

(2) Describe the dependent *simple* rings. As the referee pointed out, the Leavitt algebras $L_K(1, 1+k)$ of type $(1, 1+k)$ (see [11]), are examples of simple rings which are not dependent.

In what follows we prove that matrix rings over dependent rings are dependent. First we need the following

Lemma 10 *Let $R \subseteq S$ be nonzero rings, with S isomorphic to R^k for a finite cardinal k. Then S is dependent iff R is dependent.*

Proof To prove the "if" part, consider an inclusion $S^m \subseteq S$. This implies that $R^{mk} \subseteq R^k$. Thus, $mk \leq k$, and this implies that $m \leq 1$. The proof for the "only if" part is similar: consider an inclusion $R^m \subseteq R$. Then $R^{mk} \subseteq R^k$ and so $S^m \subseteq S$. Hence $m \leq 1$. □

Theorem 11 *For a ring R the following conditions are equivalent*

(i) R is dependent;

(ii) for every natural number n, $\mathbb{M}_n(R)$ is dependent;

(iii) for some natural number n, $\mathbb{M}_n(R)$ is dependent.

Proof (i) $\Rightarrow$ (ii) Follows from the previous lemma, since if $S = \mathbb{M}_n(R)$ then S is isomorphic to R^{n^2}.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (i) Start with $a, b \in R$ and consider $aI_n, bI_n \in \mathbb{M}_n(R)$. By hypothesis, there are $S, T \in \mathbb{M}_n(R)$, not both zero (matrix) such that $aI_nS + bI_nT = 0_n$. If any $s_{ij} \neq 0$ then $as_{ij} + bt_{ij} = 0$ shows a, b are dependent. If any $t_{ij} \neq 0$, again $as_{ij} + bt_{ij} = 0$ shows a, b are dependent. □

Remark In Exercise **17.1** [9], it was required to show that R satisfies IBN (or the rank condition) iff $\mathbb{M}_n(R)$ (for some $n \geq 1$) satisfies IBN (resp. the rank condition), but the strong rank condition was not mentioned. Recently, (private communication), T. Y. Lam provided a proof also for this missing case. The proof starts with the above lemma, has "right SRC" instead of "(right) dependent", and continues with an inclusion $S^m \subseteq S^n$. Our proof above just replaces n by 1.

Corollary 12 *Matrix rings over commutative rings are dependent.*

Proof The statement follows from the previous theorem. Here is a direct proof.

For two given matrices $A, B \in \mathbb{M}_n(R)$ over a commutative ring R, the equation $AS + BT = 0$ amounts to a homogeneous linear system with n^2 equations and $2n^2$ unknowns. It is well-known (see N. McCoy theorem, **5.3** in [4] and corollary **5.9**) that such systems have infinitely many solutions, and so also nonzero solutions. □

The previous theorem yields also a large amount of dependent matrix rings over noncommutative rings.

Corollary 13 *Let R be a ring and $S = \mathbb{M}_n(R)$ for some $n > 1$. The matrix ring S is dependent in each of the following cases:*

(i) R is a division ring; R is local; R is Boolean; R is finite;

(ii) R is right (or left) Artinian; R is unit-regular; R is right Noetherian; R has finite uniform dimension.

Proof As for the last two examples, see the remark before Proposition 25. □

For (upper) triangular matrices we can prove the following

Theorem 14 *Triangular matrix rings over dependent rings are dependent.*

Proof To simplify the writing, we discuss the $n = 2$ case, that is, for a dependent ring R we consider $A, B, S, T \in \mathbb{T}_2(R)$.

For $A = \begin{bmatrix} a & a' \\ 0 & a'' \end{bmatrix}$, $B = \begin{bmatrix} b & b' \\ 0 & b'' \end{bmatrix}$ and $S = [s_{ij}]$, $T = [t_{ij}]$ we have

$$AS + BT = \begin{bmatrix} as_{11} + bt_{11} & as_{12} + a's_{22} + bt_{12} + b't_{22} \\ 0 & a''s_{22} + b''t_{22} \end{bmatrix}.$$

By dependence of R, there exist s_{11}, t_{11} not both zero with $as_{11} + bt_{11} = 0$. Choosing zero all the other entries in S and T gives $AS + BT = 0_2$ and we cannot have both $S = T = 0_2$.

Remarks (1) This way, it also follows that $\mathbb{T}_n(R)$ is a *dependent subring of* $\mathbb{M}_n(R)$, for any dependent ring R.

(2) The special case of triangular matrices over commutative rings can also be settled using McCoy's theorem: For two given matrices $A, B \in \mathbb{T}_n(R)$ over a commutative ring R, the equation $AS + BT = 0$ amounts to a homogeneous linear system with $\left(\frac{n(n+1)}{2}\right)^2$ equations and $2\left(\frac{n(n+1)}{2}\right)^2$ unknowns.

Since we intend to use Hagany-Varadarajan results (see [7]), for convenience, we now state and prove properties for formal (lower) triangular matrix rings.

Proposition 15 *Let* $T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$ *be a formal triangular ring with* A, B *rings and* ${}_BM_A$, *a bimodule.*

(1) If B *is right dependent so is* T.

(2) If A *is left dependent so is* T.

(3) If A *is right dependent,* T *may not be right dependent.*

Proof (1) Let $\begin{bmatrix} a & 0 \\ m & b \end{bmatrix}, \begin{bmatrix} a' & 0 \\ m' & b' \end{bmatrix} \in T$. Since B is right dependent there are $s, s' \in B$ not both zero, such that $bs + b's' = 0$. Therefore $\begin{bmatrix} a & 0 \\ m & b \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} a' & 0 \\ m' & b' \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & s' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & bs + b's' \end{bmatrix} = 0$, with $\begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & s' \end{bmatrix}$ not both zero.

(2) If A is left dependent, there are $s, s' \in A$ not both zero, with $sa + s'a' = 0$. Now $\begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} + \begin{bmatrix} s' & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a' & 0 \\ m' & b' \end{bmatrix} = \begin{bmatrix} sa + s'a' & 0 \\ 0 & 0 \end{bmatrix} = 0$, with $\begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} s' & 0 \\ 0 & 0 \end{bmatrix}$ not both zero.

(3) An example is given (below) in the proof of Proposition 24. □

Remark It is easy to check that two (arbitrary) matrices are right (or left) dependent iff their transposes are left (resp. right) dependent. However, for lower triangular matrices, we cannot use this since the transpose gives an upper triangular matrix.

Next, we have another useful

Lemma 16 *Let* $a, b \in R$ *with* $aR \cap bR \neq \{0\}$. *Then* a *and* b *are dependent. If* a *and* b *are not left zero divisors, the converse also holds.*

Proof Obviously $a, b \neq 0$ and both implications follow from definitions. □

Therefore

Proposition 17 *For a domain* R *the following conditions are equivalent*

(i) R *is (right) dependent;*

(ii) R *has the right SRC;*

(iii) R *is right Ore;*

(iv) for every nonzero $a, b \in R$, $aR \cap bR \neq \{0\}$.

Proof Since (ii) ⇔ (iii) is Exercise 21 in §10, [9], and (iii) ⇔ (iv) is **10.19** [9], it suffices to check (i) ⇔ (iv).

This follows from the previous lemma since 0 is absolutely dependent in any ring. □

In closing the constructions discussion, recall that if R is a right Ore domain then so is the polynomial ring $R[x]$.

Proposition 18 *If* $R[x]$ *is dependent then* R *is dependent. The converse fails.*

Proof Let $a, b \in R$. By hypothesis, there exist $s(x) = s_0 + s_1x + ... + s_mx^m$, $t(x) = t_0 + t_1x + ... + t_nx^n$ not both zero, such that $as(x) + bt(x) = 0$. Assume (say) $m \leq n$. Then

$as_0 + bt_0 = 0$, $as_1 + bt_1 = 0$,..., $as_m + bt_m = 0$ and $bt_{m+1} = ... = bt_n = 0$. Since all coefficients s_i, t_j cannot be zero, a, b are dependent.

Conversely, in [5], it is proved that if $R[x]$ right (or left) Ore then R is Dedekind finite. Thus, for any Ore ring R (e.g., a von Neumann regular ring) that is not Dedekind finite, $R[x]$ fails to be right or left Ore. Specializing to domains and using Proposition 17, for any dependent ring that is not Dedekind finite, $R[x]$ fails to be dependent. □

Note that the failure of the converse requires a dependent ring that is not DF. Such an example is **H**, the last section.

Proposition 19 *If $R[[x]]$ is dependent so is R. The converse fails.*

Proof Similar to the previous proposition. According to Proposition 17, Kerr's example (i.e., a right Ore domain such that the power series ring $R[[x]]$ is not a right Ore domain, see [8]) is a (right) dependent domain whose power series ring is not (right) dependent. □

3 Refinements

In what follows we delimit the class of dependent rings by the class of (right) SRC rings and some class of rings (denoted IBN_1) which includes the IBN rings.

Since this will be useful for our subject, we first propose the following refinements for the SRC, the RC and the IBN properties.

Definitions For any given $n \geq 1$, we say that:

R has *right SRC$_n$* iff every $n+1$ elements of $(R^n)_R$ are dependent iff any set of independent elements of $(R^n)_R$ has cardinality $\leq n$ iff $m \leq n$ whenever an R-monomorphism $R^m \to R^n$ exists.

R has *right RC$_n$* iff every set of generators of $(R^n)_R$ has at least n elements iff $m \leq n$ whenever an R-epimorphism $R^n \to R^m$ exists.

R has *IBN$_n$* iff for every m, $(R^m)_R \cong (R^n)_R$ implies $m = n$ iff $(R^n)_R$ has rank n.

Thus *R has SRC (or RC or IBN) iff for every $n \geq 1$, R has SRC$_n$ (resp. RC$_n$, resp. IBN$_n$).*

Therefore

Proposition 20 *If R satisfies the right strong rank condition then R is (right) dependent.*

Proof As follows from the above paragraph, R is dependent iff R satisfies the right SRC_1. □

Remark Since nonzero right Noetherian rings and rings with finite uniform dimension satisfy the right SRC (see [9] **1.35** and **1.37**), these are also examples of right dependent rings.

It is easy to see that

Proposition 21 *For any $n \geq 1$ and any ring R, SRC$_{n+1} \Rightarrow$ SRC$_n$.*

Proof Suppose R does not satisfy SRC_n. There exists $m > n$ and a monomorphism $\alpha : R^m \to R^n$.

If $m > n + 1$, we get a monomorphism $R^m \to R^n \to R^{n+1}$ by composition with the canonical injection $R^n \to R^{n+1}$.

If $m = n + 1$, we get a monomorphism $R^{n+2} \to R^{n+1}$ by taking the product $\alpha \times 1_R : R^{n+1} \times R \to R^n \times R$.

In both cases, R does not satisfy SRC_{n+1}. □

Therefore,

$$... \Rightarrow SRC_{n+1} \Rightarrow SRC_n \Rightarrow ... \Rightarrow SRC_2 \Rightarrow SRC_1$$

with SRC_1 being precisely the (right) dependence condition.

Similarly,

Proposition 22 *For any $n \geq 1$ and any ring R, $RC_{n+1} \Rightarrow RC_n$.*

Therefore,

$$... \Rightarrow RC_{n+1} \Rightarrow RC_n \Rightarrow ... \Rightarrow RC_2 \Rightarrow RC_1$$

and (right) dependent rings (i.e. SRC_1 rings) satisfy RC_1.

Remarks (1) Note that a ring does not satisfy SRC_1 iff there exists an R-monomorphism $R^2 \to R$. Indeed, if there exists an R-monomorphism $R^m \to R$ for some $m > 1$, the composition with the inclusion $R^2 \to R^m$ gives an R-monomorphism $R^2 \to R$. Similar characterizations hold for each SRC_n.

(2) Analogously, a ring does not satisfy RC_1 iff there exists an R -epimorphism $R \to R^2$. Indeed, if for some $m > 1$ there exists an R-epimorphism $R \to R^m$, the composition with the projection of R^m onto R^2 gives an R-epimorphism $R \to R^2$. Similar characterizations hold for each RC_n.

While $\mathrm{(S)RC}_k \Rightarrow \mathrm{(S)RC}_l$ whenever $k \leq l$, this fails for the IBN refinements.

Proposition 23 *If $k \leq l$, $IBN_k \Rightarrow IBN_l$ fails.*

Proof (G. Bergman). From [2], it follows that given any congruence C on the additive semigroup of positive integers, there is a ring R such that $R^m \cong R^n \Leftrightarrow (m, n) \in C$ (where $\cong$ means isomorphism as left, equivalently, as right modules). So for every k, there is a ring R for which $R^m \cong R^n$ iff either $m = n$, or m and n are both $> k$. □

Hence, none of $\mathrm{IBN}_1 \Rightarrow \mathrm{IBN}_2$, $\mathrm{IBN}_2 \Rightarrow \mathrm{IBN}_3$, and so on, holds.

Note that, for any $n > 0$, the class of rings which have IBN_n includes the class of rings which have IBN. Rephrasing, the IBN property is the (logical) conjunction of all the IBN_n properties. As such, IBN implies every IBN_n, for any $n > 0$.

Proposition 24 *Dependent rings have IBN_1, that is, have a rank and this rank is 1. The converse fails.*

Proof In any ring R, $\{1\}$ is a basis for R_R (or ${}_RR$). In a dependent ring, independent sets with at least two elements, do not exist, so dependent rings have a rank, and this is 1. Hence every dependent ring has IBN_1, which is larger than IBN.

Conversely, in an IBN ring, independent elements may exist which do not span the whole ring, that is, IBN rings may not be dependent. For an example we take advantage of results obtained in [7]. In turn, they use an example obtained by J. Kerr (1982, [8]), that is, a right Ore domain R with $R[[X]]$ not right Ore. For this Ore domain, in [7] it is proved (Proposition 4.4) that $T = \begin{bmatrix} R & 0 \\ R[[X]] & R[[X]] \end{bmatrix}$ does not satisfy the right SRC.

In the proof, a pair (α, β) with $\beta : R[[X]]^2 \to R[[X]]$ injective in Mod-$R[[X]]$, $\alpha : R^2 \oplus R[[X]]^2 \to R \oplus R[[X]]$ injective in Mod-R, gives rise to an injective map $T^2 \to T$ in Mod-T. This shows that T does not satisfy the right SRC_1 in Mod-T, so T is not right dependent. However, according to Proposition 4.1 (see [7]), T has IBN iff one of A, B has it. Hence, it follows that the T constructed above has IBN (and so also IBN_1) but is not right dependent. □

Question Are dependent rings also IBN ? In particular, are dependent rings, IBN_2 ? A positive answer amounts to: $\mathrm{SRC}_1 \Rightarrow \mathrm{IBN}_2$.

As the referee pointed out, by Remark 3.17 of [1], the Leavitt path algebra $L_K(E)$ of a finite graph E over a field K has IBN iff $L_K(E)$ has IBN_1. Then, by the previous proposition, we obtain that if $L_K(E)$ is dependent, then $L_K(E)$ has IBN.

Moreover, if $L_K(E)$, which was introduced in Example 3.8 of [12], is dependent, this would give a negative answer for the Question.

Proposition 25 *For every n, SRC_n implies RC_n.*

Proof By the universal property of the free R-module R^n, every epimorphism $\alpha : R^n \to R^m$ splits, i.e., there is a monomorphism $\beta : R^m \to R^n$ such that $\alpha \circ \beta = 1_{R^n}$. By SRC_n, $m \leq n$. □

Proposition 26 *For every n, RC_n implies IBN_n.*

Proof Suppose a ring R has not IBN_n. There exists an isomorphism $\alpha : R^m \to R^n$ with $m \neq n$. If $m > n$, $\alpha^{-1} : R^n \to R^m$ is (also) an epimorphism and so R does not satisfy RC_n. If $m < n$, $\alpha : R^m \to R^n$ is (also) an epimorphism. Then $\alpha \times 1_{R^{n-m}} : R^n \to R^{2n-m}$ is an epimorphism and since $2n - m > n$, R does not satisfy RC_n. □

Corollary 27 *For every n, SRC_n implies IBN_n.*

For $n = 1$, this gives an alternative proof for the first statement in Proposition 24.

Hence so far we have {rings with the right SRC} $\subseteq$ {right dependent rings} $\subseteq$ {rings with IBN_1}.

The *right inclusion is strict* by the Proposition 24**.** To give an example which shows that the *left inclusion is strict* (that is, with SRC_1 but not SRC_2) seems to be harder. This example would be a part of a larger project which is not addressed here: to show (by suitable examples) that all the refinements introduced in this section are different.

4 Special examples

We start with a construction (G. Bergman) which will be useful.

Proposition 28 *Let A be a ring which is not dependent and let B be an over-ring such that every pair of elements of A is dependent over B; i.e., for all $a, a' \in A$, there are $b, b' \in B$, not both zero, such that $ab + a'b' = 0$. Adjoin to B a central square-zero element ε, and, within $B[\varepsilon]$, consider the subring $R = A + \varepsilon B$. Then R is a dependent ring.*

Proof Let $x_1 = a_1 + \varepsilon b_1$ and $x_2 = a_2 + \varepsilon b_2$ ($a_1, a_2 \in A, b_1, b_2 \in B$) be two elements of R. If $a_1 = 0$, then we have the linear relation $x_1\varepsilon + x_2 0 = 0$, and similarly if $a_2 = 0$. If a_1 and a_2 are both nonzero, then by hypothesis there is a nontrivial relation $a_1b_1 + a_2b_2 = 0$ in B. This yields the relation $a_1(\varepsilon b_1) + a_2(\varepsilon b_2) = 0$ and so $x_1(\varepsilon b_1) + x_2(\varepsilon b_2) = 0$ in R. □

Note that this construction shows that *every not dependent ring can be embedded in a dependent ring*.

With some exceptions (direct products, formal matrix rings, polynomial rings and power series rings were discussed in the previous section), dependent rings behave badly with respect to Ring Theory constructions.

This is (mainly) because ring homomorphisms do not preserve nor reflect dependence. Neither do surjective (or injective) ring homomorphisms.

A. *A subring of a dependent ring need not be dependent.*

Clearly, two elements could be independent in a *subring* (having "less scalars") but dependent in the whole ring. So subrings of dependent rings may not be dependent. For an example we just use the previous proposition: R is dependent but its subring A is not.

Corners of (unit) regular rings are (unit) regular. In Corollary 3. 12 of [12] it was showed that a corner of a IBN ring may not have IBN.

In Theorem 3.9 of [1], $L_K(R)$ has RC. This shows that corners of rings with RC have no RC in general.

There seem to be no results about corners of rings with right (or left) SRC.

Question Are corners of dependent rings, also dependent ?

Hint. As the referee pointed out, the following example could answer the question in the negative. Let K be a field and E the following graph:

$$E = \overset{\curvearrowright}{\bullet^{v}} \longrightarrow \underset{\circlearrowleft}{\overset{\curvearrowright}{\bullet^{u}}}$$

We then have that the corner $uL_k(E)u$ is isomorphic to the Leavitt algebra $L_K(1, 2)$, and so $uL_K(E)u$ is neither dependent nor has RC. If $L_K(E)$ is dependent then this shows that corners of dependent rings may not be dependent. This would also answer in the negative the

Question Are dependent rings Morita invariant?

Note that IBN is not a Morita invariant property (see Example 11, p. 502, [9]).

B. *A dependent ideal of an independent ring.*

We have already mentioned that if D is a division ring, $V = \bigoplus_{i=1}^{\infty} e_i D$ and $E = \mathrm{End}(V_D)$, then E is not dependent (having independent endomorphisms). It is (maybe) less known that E has only one nontrivial (i.e. $\neq 0$ and $\neq E$) ideal (see Exercise **3.15** [9]): the set F of all finite rank endomorphisms. That is, $F = \{f \in E : \dim_D f(V) < \infty\}$ is (also) the unique maximal ideal of E.

We can show that F is dependent. Suppose $f, g \in E$, $\dim_D(f(V)) = m$ and $\dim_D(g(V)) = n$ and $m \leq n$. We can identify f, g with some $n \times n$ matrices over D. As such these are dependent (by Corollary 13).

C. *Ring monomorphisms do not reflect dependence* (i.e., if $f : R \to R'$ is a ring monomorphism and R' is dependent, R may not be dependent).

In Proposition 28, just consider the inclusion $i : A \to R$. There R is dependent but A is not.

D. *Ring epimorphisms do not reflect dependence*. Equivalently, *homomorphic images (and so quotient rings) of independent rings may be dependent.*

Actually, clearly, two elements could be dependent in a ring but their cosets in a *quotient ring* could be independent (again, having "less scalars").

For an example, consider (**1.31** [9]) R, the free algebra $k\langle X\rangle$ generated over a field k by a set X with $|X| \geq 2$. If $x \neq y$ in X, then in the right regular module R_R the elements $\{u_j = x^j y : 0 \leq j < \infty\}$ are (right) independent. We can map R onto k, and k is dependent but R is not.

E. For any ring R, *the property "R is dependent" and the property "R is regular", are (logically) independent.*

The next three examples are justified since unit-regular rings are dependent and regular.

(a) *Regular rings need not be dependent*.

An example of regular ring that is not dependent is well-known from long time ago (see [3], Exercise 8, 1955): the ring E in example **B** is (von Neumann) regular and includes independent linear transformations.

(b) *Dependent rings need not be regular*.

It suffices to give an example of commutative ring which is not regular. Since regular rings are reduced, for any prime p, $\mathbb{Z}(p^2)$ is not regular.

As for an infinite example, for any field k, we can take $k[x]/x^2$.

As a matrix ring example, if R is a commutative ring which is not regular and $n \geq 1$ then, by Proposition 12, $\mathbb{M}_n(R)$ is dependent but not regular

(c) *A regular dependent ring need not be unit-regular.*

For any field k consider the triangular matrix ring $\mathbb{T}_2(k) = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$. According to Proposition 14, $\mathbb{T}_2(k)$ is dependent and since k is regular, so is $\mathbb{T}_2(k)$. However, $\mathbb{T}_2(k)$ is not unit-regular. Indeed $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not unit-regular since $A\mathbb{T}_2(k)A = 0_2$.

F. *An example of left dependent ring which is not right dependent*.

It is well-known that the IBN property is left-right symmetric but the strong rank condition is not. The example used in order to show, via left and right Ore domains, that the strong rank condition for rings is not left-right symmetric (see second paragraph of **10C**) is also suitable for our purpose.

Using twisted polynomial rings, let σ be an endomorphism of a division ring R and $S = R[x; \sigma]$. Then S is left Ore and, by Proposition 17, is also left dependent.

If $\sigma(R) \neq R$, say $t \in R \diagdown \sigma(R)$ then $\{1, t\}$ are right (linearly) independent over $\sigma(R)$. Then $\{x, tx\}$ are right (linearly) independent over S, so S is not right Ore nor right dependent

.

G. *Right (or left) dependent rings need not be clean, nor exchange*.

Such an example is justified since unit-regular rings are clean and (so) exchange.

However, $\mathbb{Z}$ is not exchange, but being commutative, is left and right dependent.

In contrast with unit-regular rings, commutative rings and right (or left) Noetherian rings which all are Dedekind finite, here is

H. *A dependent ring which is not Dedekind finite*.

We use Shepherdson's (see [13]) example of domain (and so DF ring) R such that $\mathbb{M}_2(R)$ is not DF. Since according to Theorem 11, $\mathbb{M}_2(R)$ is dependent iff R is dependent, it only remains to check that this domain R, is dependent. Equivalently (by Proposition 17), to verify $aR \cap bR \neq \{0\}$, for every nonzero $a, b \in R$.

This can be done using bringing the elements of R to a "normal form". To have all details, we direct the reader to the solution of Exercise **1.18**, [10].

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Declarations

Conflict of interest In my manuscript, there is no conflict of interests.

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