CONJUGATION RELATIONS IN γ-CATEGORIES

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Introduction. Considering the theory of abelian categories as "the" natural generalization of the category of abelian groups, one can ask himself which is "a" corresponding generalization of the category of non-necessary abelian groups.

More precisely, such a generalization is required to have some "canonic" properties: to be a conormal but not a normal category (in this way we have to distinguish exact and coexact sequences), to be complete, cocomplete, etc.

In such a generalization one would expect to prove and use Noether, Schreier, Jordan-Hölder theorems and "5", "9", and Zassenhaus lemmas.

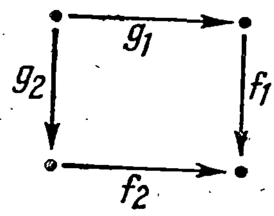
There are plenty of such generalizations in the literature, two of these being the "hofmanian" and the "γ-categories", the last ones being introduced by Burgin and Calenko.

In such categories, having about all the "working" theorems, one can expect to give some kind of theory of formations in the Gaschütz sense. A difficulty which appears at once is the fact that conjugation cannot be defined globally (without elements); the author uses an abstract conjugation relation proposed by Schunck which permits an easy approach to the subject.

Let $\mathcal{A}$ be an arbitrary γ-category in Burgin-Calenko's sense [1, 2] (for a more didactical exposition of these categories see [4]).

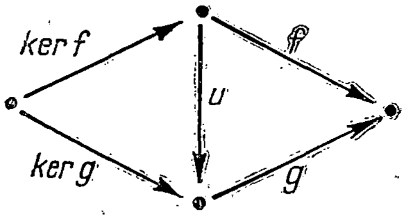
First, let us recall that a γ-category is a locally small, conormal category with zero and epi-mono factorizations, satisfying the following three conditions:

γ1. $\mathcal{A}$ has pullbacks of two morphisms, provided that at least one of them, f_1 is mono and if f_2 is epi then so is g_1.



γ2. If f is a normal mono and g is epi then $im(gf)$ is a normal mono.

γ3. In the following commutative diagram if f, g are epis and u is mono then u is iso.



One shows that a γ-category has kernels, cokernels for normal monos, images, coimages, inverse images, finite intersections, that any morphism f factors through *coker* (*ker*(f)) ans *im*(f) and that a γ-category is balanced.

Essentially for what follows is that a γ-category has unions of two subobjects provided that at least one of them is normal and that in a γ-category the 5-lemma, the 9-lemma and the Noether isomorphism theorems hold, the two last ones in the following form:

N1. If $u : M \to A$ and $v : N \to A$ are normal monos and $v \leqslant u$ then $A/N/M/N \cong A/M$.

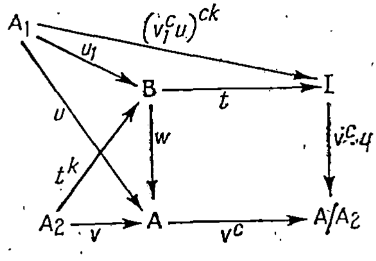
N2. If $u : A_1 \to A$ and $v : A_2 \to A$ are monos, the second being normal, then $A_1/A_1 \cap A_2 \cong A_2 \cup A_1/A_2$.

We shall simplify the exposition using, from now on, equalities instead of isomorphisms, which means in fact that we shall work in a skeletal category $\mathfrak{A}$.

Finally, let us recall also from [4] (for instance) the following two lemmas:

L1. If $vu = ker(w)$ and v is mono then $u = ker(wv)$.

L2. In the conditions of N2, the pullback of $coker(v)$ and $im(coker(v).u)$ over A/A_2 is the union $A_1 \cup A_2$.



Using the diagram describing this last lemma, where $I = im(coker(v).u)$, one easily shows that (as objects!) $im(v^c \cdot w) = (v^c \cdot w)^I = v^{ck} = I$ where the obvious notations are taken also from [4]. Consequently, we have $im(A_1 \cup A_2 \to A \to A/A_2) = im(A_1 \to A_1 \cup A_2 \to A \to A/A_2)$.

Following H. Schunck [6] we shall use the following definitions:

Definition 1. A binary relation on obj $\mathfrak{A}$, h is called a *conjugation relation* if

(0) A_1hA implies $A_1 \subseteq A$
(1) A_1hA and AhA implies $A_1 = A$
(2) $A_1 \subseteq A_2 \subseteq A$ and A_1hA implies A_1hA_2
(3) A_1hA implies $(f/A_1)h(f/A)$ for every morphism f of domain A.
In this case A_1 is called *h-subobject of A*.

DEFINITION 2. A full subcategory $\mathfrak{X}$ of $\mathfrak{A}$ is called *homomorph* if $\mathfrak{X}$ is closed under (epimorphic) images of morphisms of $\mathfrak{A}$ with domain in $\mathfrak{X}$.

DEFINITION 3. A subobject $m_1 : A_1 \to A$ is called a *$\mathfrak{X}$-covering subobject* if

(i) $A_1 \in \text{obj}\ \mathfrak{X}$.
(ii) For each subobject $m_2 : A_2 \to A$ such that $m_1 \leqslant m_2$ and each normal subobject $A' \to A_2$, if $A_2/A' \in \mathfrak{X}$ then $A' \bigcup A_1 = A_2$.

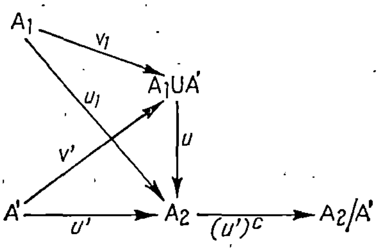
First of all, it is trivial that

THEOREM 1. *If A_1 is a $\mathfrak{X}$-covering subobject of $\mathfrak{A}$ and $A_1 \subseteq A_2 \subseteq A$, then A_1 is $\mathfrak{X}$-covering for A_2.*

THEOREM 2. *If h is a conjugation relation in $\mathfrak{A}$, then there is a homomorph $\mathfrak{X}$ in $\mathfrak{A}$ such that each h-subobject of A is $\mathfrak{X}$-covering for A.*

Proof. Let $\mathfrak{X}$ be the full subcategory of $\mathfrak{A}$, consisting of all objects A such that AhA. Letting $A_1 = A$ in (3), it is readily seen that $\mathfrak{X}$ is a homomorph.

Now let A_1 be a h-subobject of A. Letting $A_2 = A_1$ in (2) we have $A_1 \in \mathfrak{X}$. Next, in order to verify (ii) for A_1, suppose $A_1 \subseteq A_2 \subseteq A$, A' normal subobject of A_2 and $A_2/A' \in \mathfrak{X}$. The following diagram describes our situation where $u' = ker((u')^c)$ and $(u')^c = coker(u')$.

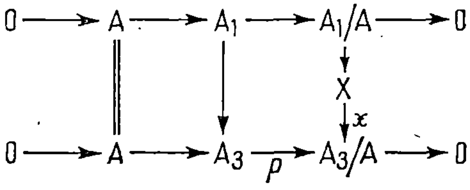


Using L1 we have $v' = ker((u')^c \cdot u) = im((u')^c \cdot u_1) = (u')^c/A_1$, where the middle equality uses L2.

Since from (2) we already have A_1hA_2, letting $A = A_2$ and $f = (u')^c$ in (3), we obtain $((u')^c/A_1)h((u')\cdot(A_2))$ so that $(A' \bigcup A_1/A')h(A_2/A')$. But by assumption $A_2/A' \in \mathfrak{X}$, hence $(A_2/A')h(A_2/A')$. A final application of (1) gives us $A' \bigcup A_1/A' = A_2/A'$ and then using for instance the 5-lemma we obtain $A' \bigcup A_1 = A_2$.

For the proof of the the third and last theorem we need the following two lemmas:

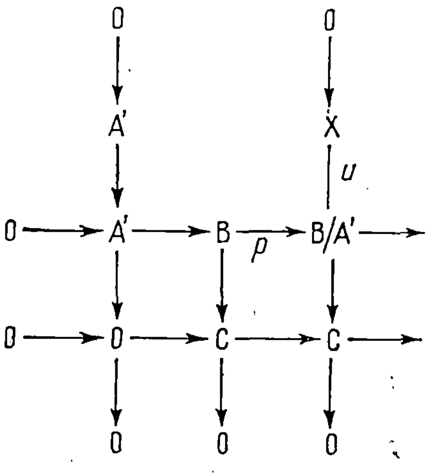
LEMMA 3. *If the following diagram has exact rows then there is a subobject $m_2: A_2 \to A_3$ with $m_1 \leqslant m_2$ such that $X = A_2/A$; of course A being normal subobject of A_3, so will be A in A_2 too.*



Proof. A_2 is the pullback of p and x so that $m_1 \leqslant m_2$ is then immediat. It is also easy to show that $A \to A_1 \to A_2 = ker(A_2 \to X)$ and hence, $A_2 \to X$ being epi, it follows by $\gamma 1$ that $X = A_2/A$.

LEMMA 4. *If $X \to B/A'$ is a normal subobject then there is a normal subobject $B' \to B$ such that $X = B'/A'$.*

Proof. If $C = coker(X \to B/A')$, we have the following diagram with exact rows and columns



Hence, if B' is the pullback of p and u, it is easy to show that $B' \to B = ker(B \to C)$. Using the 9-lemma it follows that $X = B'/A'$.

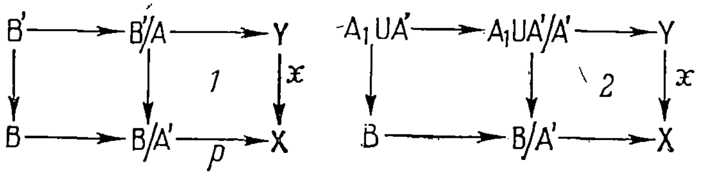
THEOREM 5. *If A_1 is $\mathfrak{X}$-covering for A and $A' \to A$ is a normal subobject, then $A' \cup A_1/A'$ is $\mathfrak{X}$-covering for A/A'.*

Proof. Using U2 we have $A' \cup A_1/A' = A_1/A' \cap A_1 \in \mathfrak{X}$, because $\mathfrak{X}$ is homomorph and $A_1/A' \cap A_1$ is an epimorphic image of $A_1 \in \mathfrak{X}$, and so (i) is verified.

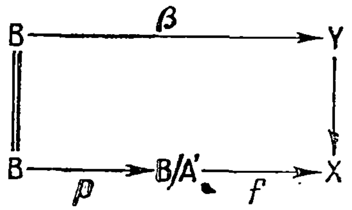
Using the two previous lemmas, in order to verify the condition (ii), we suppose $A_1 \cup A'/A' \subseteq B/A' \subseteq A/A'$ and B'/A' normal subobject of B/A' such that $B/A'/B'/A' \in \mathfrak{X}$, and we have to show that $(B'/A')_1 \cup \cup (A_1 \cup A'/A') = B/A'$.

In these conditions we have $A_1 \subseteq B \subseteq A$ and B' normal subobject of B. Using N1, $B/B' = B/A'/B'/A' \in \mathfrak{X}$ and hence $B' \cup A_1 = B$, because A_1 is $\mathfrak{X}$-covering for A. Further, we have $B' \cup (A_1 \cup A') = B$, because A' is a normal subobject of B'.

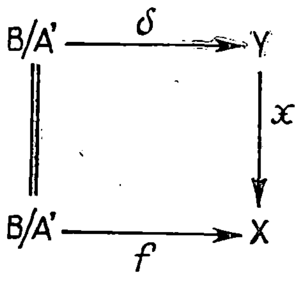
Now let $f: B/A' \to X$ be a morphism and $x: X \to Y$ be a mono in $\mathcal{C}$ such that squares 1,2 in the following diagrams are commutative.



Using the commutativity of the outer rectangles (filled up in a canonic way) and equality $B = B' \cup (A_1 \cup A')$ we get a morphism $\beta: B \to Y$ which makes the following diagram commutative.



If finally $k: A' \to B$ is the kernel of p, then p is the cokernel of k and from $fpk = x\beta k = 0$ or $\beta k = 0$, we get a morphism $\delta: B/A' \to Y$ such that $\beta = \delta p$. Hence, p being epi, the following diagram is commutative which proves that $B/A' = (B'A') \cup (A_1 \cup A'/A')$, q.e.d.



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RELAŢII DE CONJUGARE ÎN γ-CATEGORII

(Rezumat)

Inspirîndu-se din teoria formaţiilor (în sensul lui Gaschütz) a grupurilor resolubile finite şi utilizînd ideea mai generală (datorită lui H. Schunck) de a considera relaţii de conjugare în locul relaţiilor de 𝔛-maximalitate sau 𝔛-proiector, autorul dă un punct de plecare posibil în teoria formaţiilor într-o γ-categorie.

Se scoate în acelaşi timp în evidenţă asemănarea dintre γ-categorii şi *Gr*, categoria grupurilor (non-necesar abeliene).

СООТНОШЕНИЯ СОПРЯЖЕНИЯ В γ-КАТЕГОРИЯХ

(Резюме)

Исходя из теории формаций (в смысле Гашюца) конечных разрешаемых групп и используя более общую идею (принадлежащую Германну Шунку) рассматривать соотношения сопряжения вместо соотношений 𝔛-максимальности или 𝔛-проектора, автор дает возможную исходную точку в теории формаций в γ-гатегории.

Выявляется одновременно сходство между γ-категориями и Gr, категорией групп (не необходимо абелевых).