COCOMPACT ELEMENTS IN ALGEBRAIC LATTICES

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Let L be a complete lattice. A subset X of L is called a <u>discover</u> of an element a of L if $\bigwedge X \leq a$. An element a from L is called <u>cocompact</u> if each discover of a has a finite subdiscover, i.e. if $\bigwedge X \leq a$ implies the existence of a finite $X_f \subseteq X$ such that $\bigwedge X_f \leq a$.

Dualising well-known lattice properties of compact elements in lattices one can easily prove the following ones:

(A) the meet of two cocompact elements is a cocompact element too; an element greater than a cocompact element is generally not cocompact, so that cocompact elements do not form a filter.

(B) a complete lattice is artinian if and only if each element is cocompact.

(C) in a lower continuous lattice every artinian element is cocompact.

The author has considered that a reasonable plan of research of cocompact elements in lattices that occur in algebra (and are algebraic too !) would be the following:

I. Cocompact elements in the lattice of subsets of a given set with respect to set-theoretic inclusion.

II. Cocompact elements in the lattice of subspaces of a given vector space over a division ring.

III. Cocompact elements in the lattice of subgroups of a given abelian group.

IV. Cocompact elements in the lattice of submodules of a given module over an arbitrary ring.

V. Cocompact elements in the lattice of subalgebras of a given universal algebra.

In what follows we describe the complete solution of the first and second case and give some results concerning the third case. Meanwhile the cases IV and V seem to be extremely difficult.

I. Theorem 1.- Let A be a set and B a subset of A. In the lattice of the subsets of A, ordered by inclusion, B is a cocompact element if and only if A - B is finite.

The *proof* is elementary and reduces to the following more general facts:

(i) in a bounded distributive lattice if the elements a and b have complements and $a \leq b$ then $b' \leq a'$.

(ii) each Boolean complete lattice is infinitely distributive and satisfies the De Morgan's identities.

So, we can derive a more general result: in a Boolean complete lattice an element is cocompact if and only if its complement is compact. Hence, the compact elements in the lattice of subsets of A being the finite subsets, the theorem is proved.

Remark.- The generalization of the theorem provides only a small step: if someone adds the condition of atomicity(each element $a > 0$ contains an atom in the sublattice $a/0$) the lattice becomes completely distributive and so isomorphic to the Boolean algebra of all subsets of a set.

Conclusion.- Let A be a finite set. Each element in the lattice of all the subsets of A are cocompact and compact. If A is infinite, the finite subsets of A are the compact elements and the subsets with finite complement are the cocompact elements in the lattice of all the subsets of A (so that there are not compact-cocompact elements).

II. Theorem 2.- Let V be a vector space of finite dimension . In the lattice of all subspaces of V every element is compact and cocompact.

Proof. We first observe that the dual of (B) is of course true and we record it as

(D) a complete lattice is noetherian if and only if each element is compact. We also mention

(E) a modular complemented lattice is noetherian if and only if it is artinian.

The lattice L(V) of all the subspaces of V is modular and complemented. Its compact elements are the subspaces of finite dimension (more general, the compact subalgebras in the lattice of all subalgebras of an universal algebra are the finite generated ones).
If V is a finite dimensional vector space all its subspaces are compact elements so that L(V) is noetherian according to (D). Moreover, L(V) is also artinian, using (E) so that the theorem follows from (B) and (D).

<u>Theorem 3</u>.- Let V be an infinite dimensional vector space . The latti L(V) of all the subspaces of V contains no other cocompac elemens but V itself.

Proof. Let S be a proper subspace of V and T a direct complement for
We distinguish two cases: if S is of infinite dimension, let $\{a_n / n \in \mathbb{N}$ an independent subset of S and $a \in T$, $a \neq 0$. The set $\{a + a_n / n \in \mathbb{N}\}$ is also independent so that we can consider the following decreasing sequence $S_n = \bigoplus_{m \geq n} \langle a + a_m \rangle$, $S_0 \supset S_1 \supset S_2 \supset \ldots \supset S_n \supset \ldots$, $\bigcap_{n \in \mathbb{N}} S_n = 0$.
Hence $\bigcap_{n \in \mathbb{N}} S_n \subseteq S$, but for each finite subset $F \subset \mathbb{N}$ we have $\bigcap_{n \in F} S_n =$ $= S_{\max F} \nsubseteq S$ because $S_n \subseteq S$ would imply $a \in S$. So S is not a cocompact element in L(V). If S is of finite dimension, T is infinite dimensional and let $\{a_n / n \in \mathbb{N}\}$ an independent set in T. We define the followin decreasing sequence of subspaces $S_n = \bigoplus_{m \geq n} \langle a_m \rangle$. This sequence has all the above properties so that the same argument implies that S is not a cocompact element in L(V). Indeed, $S_n \subseteq S$ would imply $S_n \subseteq S \cap T =$ and then $S_n = 0$.

<u>Conclusion</u>.- If V is a finite dimensional vector space, all the sub-

spaces are compact and cocompact elements in L(V). If V is an infinite dimensional vector space, the finite dimensional subspaces are the compact elements of L(V) and the only cocompact element of L(V) is V itself.

III. We first record some immediate consequences of the above mentioned or proved properties.

1. Recall that an abelian group is called cocyclic if it is isomorphic with $\mathbb{Z}(p^k)$ for a prime p and K = 1,2,..,∞, $\mathbb{Z}(p^\infty)$ being the quasicyclic group of Prüfer. We know that for an abelian group A , the lattice L(A) of all subgroups of A is artinian if and only if A is a finite direct sum of cocyclic groups. Using (B) we then derive that each subgroup of a finitely cogenerated group (a finite direct sum of cocyclic groups) is a cocompact element in its lattice of subgroups (direct proof ?).

- An abelian group is called reduced if it has no nontrivial divisible subgroups. Since every divisible abelian group is a direct sum of quasicyclic groups (the torsion part) and of $\mathbb{Q}$ (the torsion-free part) we deduce that the quasicyclic subgroups of each nonreduced finite direct sum of cocyclic groups are cocompact elements which are not compact (not being finitely generated). Hence, in the case of abelian groups the problem of determination of the cocompact elements in L(A) is no more trivial, such as it is in the case of vector spaces.

2. An abelian group is called elementary if it coincides with its socle i.e. is a torsion group with square-free order elements.

It is also known that each elementary p-group has a natural structure of vector space over $\mathbb{Z}(p)$. Hence, the subgroups of an elementary p-group of finite rank are compact-cocompact elements in the lattice of all subgroups but the proper subgroups of an elementary p-group of infinite rank are not cocompact elements in the corresponding lattice of subgroups.

3. Every torsion-free divisible abelian group is a direct sum of $\mathbb{Q}$ and has a natural structure of $\mathbb{Q}$-vector space. Hence, in a torsion-free divisible abelian group of finite rank all the subgroups are compact-cocompact elements in the lattice of all the subgroups; the proper divisible subgroups of a torsion-free divisible abelian group of infinite rank are not coconpact elements.

In what follows we deal with an abelian group of infinite rank and discuss several aspects concerning cocompact elements in the latti of all the subgroups.

Let A be an abelian group of infinite rank and B a proper subgroup of A.

Proposition 4.- Each subgroup of finite rank of an abelian group of infinite rank is not a cocompact elemnt in L(A).

Proof. Let K be a maximal independent system in B and $K \cup L$ a maximal independent system in A such that $K \cap L = \emptyset$ (K is also independent in A and can be extended to a maximal one in A). If $\{a_n / n \in \mathbb{N}\}$ is a countable subset of L (L is of course infinite) then the subgroups $B_n = \bigoplus_{m \geq n} \langle a_m \rangle$ form a decreasing sequence such that $\bigcap_{n \in \mathbb{N}} B_n = 0 \subseteq B$. Obviously, for every finite subset $F \subset \mathbb{N}$ we have $\bigcap_{n \in F} B_n = B_{\max F} \nsubseteq B$ because $B_n \subseteq B$ would imply $a_n \in B$ and so $K \cup \{a_n\}$ would be independent in B, contradicting the maximality of K.

Proposition 5.- If A is an abelian group of infinite rank the followi infinite torsion-free rank subgroups B are not cocomp elements in L(A): (a) $T(B) \neq T(A)$, and, (b) $r_0(B) < r_0($

Proof. Let $\{a_n / n \in \mathbb{N}\}$ an independent system of infinite order element in B. We extend this system to K, maximal independent in B and to $K \cup L$ maximal independent in A (with $K \cap L = \emptyset$). In the case (a) we consider $a \in T(A) - B$ and in the case (b), $a \in L$. Now, the systems $\{a\} \cup \{a_n / n \in \mathbb{N}\}$ and respectively $\{a + a_n / n \in \mathbb{N}\}$ are independent too, so that we can again consider the decreasing sequence $B_n = \bigoplus_{m \geq n} \langle a + a_m \rangle$ This sequence has the above mentioned properties so we again deduce that B is not a cocompact element in L(A).

Conclusions.- According to propositions 4 and 5 we deduce that the only infinite torsion-free rank subgroups who could be cocompact elements in L(A) would be the subgroups such that T(B) = T(A) and $r_o(B) = r_o(A)$. Such subgroups are essential subgroups in A (the lattice of all the subspaces of a vector space being complemented contains no essential elements).

- One case remains open, the one of the finite torsion-free rank subgroups , i.e. subgroups B such that

(i) there is a prime p and an infinite independent system in the socle $B[p] = S(B_p)$ of the p-component of B;

(ii) there is an infinite sequence of primes $\{p_1, p_2, \ldots, p_n, \ldots\}$ and an independent system $\{b_n / n \in \mathbb{N}\}$ with $ord(b_n) = p_n$.

We cannot use the algorithm above for noncocompactity because the system $\{a + a_n / n \in \mathbb{N}\}$ is no more necessarily independent even if a is an infinite order element, so that the intersection of the decreasing sequence B_n can be nonzero.

In a second part of this work we deal also with the case of the finite rank abelian groups.

References.- P.Crawley, Dilworth R.P., Algebraic theory of lattices 1973, Prentice Hall, Englewood Cliffs, New Jersey