COATOMIC LATTICES AND RELATED ABELIAN GROUP TOPICS

Grigore Călugăreanu
Dept.of Algebra
Faculty of Mathematics-Informatics
"Babeş-Bolyai" University
Cluj-Napoca, Romania

Abstract

The interest for coatomic lattices comes back to H.Bass [1] (1960) who defined B-objects, i.e. modules M such that every submodule $N \neq M$ is contained in a maximal submodule.

Several authors (Stenström[9] and Crawley-Dilworth [6] being the first ones) gave latticial generalizations of very much well-known results in module theory, lattice theory being the natural general setting for these results.

In what follows, we relate coatomic lattices with the radical, the superfluous elements and study the particular case of the abelian groups. Two classes , one dual to the cocyclic groups and the other, the groups having exactly one maximal subgroup, are characterized.

1 Latticial Results.

$\mathbb{Z}$

In what follows we shall use four similar (and dual) conditions: a lattice L is called **atomic** if for every $0 \neq a \in L$ the sublattice $a/0$ contains atoms, **coatomic** if for every $a \neq 1$,$1/a$ contains maximal elements, **reduced** if for every $a \neq 0$, $a/0$ contains maximal elements and **torsion (semiartinian)** if for every $a \neq 1$, $1/a$ contains atoms . The lattice L satisfies the condition (**R3**) if for every $a \neq 1$, a essential in L, $1/a$ contains atoms. For all the latticial notions and notation we refer to [6]. All the applications concern abelian groups.

Proposition 1.1 *(Krull)-Every compact lattice is coatomic.*□

Proposition 1.2 *Let L be a modular lattice and $a \in L$. If $a/0$ and $1/a$ are coatomic then L is coatomic.*

Proof. Let $c \neq 1$. We distinguish two cases: if $a \vee c \neq 1$ the sublattice $1/a$ being coatomic there is an element m maximal in $1/a$ and hence maximal in L such that $c \leq a \vee c \leq m$. If $a \vee c = 1$ then $a \wedge c \neq a$ (otherwise $a \wedge c = a \Leftrightarrow a \leq c \Leftrightarrow a \vee c = c = 1$) and $a/0$ being coatomic there is an element m maximal in $a/0$ such that $a \wedge c \leq m \neq a$. Using the modularity of L we verify that $m \vee c$ is maximal in L and so L is coatomic. Indeed, from $(m \vee c) \wedge a = m \vee (c \wedge a) = m$ we have $a/m = a/(a \wedge (m \vee c))$ $\cong (a \vee (m \vee c))/(m \vee c) = 1/(m \vee c)$ so that $m \vee c$ is maximal in L.□

Remark 1.1 *If L is coatomic and $a \in L$ then $1/a$ is also coatomic but generally $a/0$ is not.*

Indeed, if $a/0$ is not coatomic and we adjoin a greatest element 1, $1/0 = L$ becomes coatomic ($L \setminus \{1\} = a/0$ has a greatest element) - see also Proposition 6.

Definition 1.1 *(Benabdallah,Piche[2]) A complete lattice L is called* **reducible** *if its socle $s(L) = 1$ and* **inductive** *if for each $a \in L$ and every chain $\{b_i\}_{i\in I}$, $\forall i \in I$,$a \wedge b_i = 0 \Longrightarrow a \wedge (\bigvee_{i\in I} b_i) = 0$ and each factor sublattice (interval) of L has this property.*

Observe that a modular, reducible and inductive lattice is complemented and atomic (e.g.Theorem 9.2 [2]).

Definition 1.2 *We say that a complemented lattice L* **has enough complements** *if for every $a, b \in L$ such that $a \wedge b = 0$, a admits a complement $c \in L$ such that $b \leq c$.*

Lemma 1.1 *Every modular, atomic, complemented and inductive lattice has enough complements.*

Proof. Let $D = \{x \in L \mid b \leq x, a \wedge x = 0\}$. $D \neq \emptyset$ because $b \in D$ and so, applying Zorn's Lemma, let c be maximal in D (in fact , L inductive $\Longrightarrow$ L pseudo-complemented). If $a \vee c \neq 1$, L being complemented, let d be a complement of $a \vee c$. We have $d \neq 0$ (otherwise $(a \vee c) \vee d = 1$ implies $a \vee c = 1$) so, L being atomic, there is an atom $s \leq d$. Hence $(a \vee c) \wedge s = 0$ and from $a \wedge c = 0$ one has ([2] lemma 2.2) $a \wedge (c \vee s) = 0$, which contradicts the maximality of c. So c is a complement of a , $b \leq c$ and L has enough complements.$\Box$

Lemma 1.2 *Every modular, reducible and inductive lattice has enough complements.*

Proof. Let $\{u_i\}_{i \in I}$ be the set of all the atoms of L, $J = \{i \in I \mid u_i \leq a\}$, $T = \{i \in I \mid u_i \leq b\}$ and $c = \bigvee_{i \in J} u_i, d = \bigvee_{i \in T} u_i$. Hence $c = s(a/0)$ and $c/0$ being inductive, c is a direct sum of atoms ([2] theorem 8.6: in every inductive lattice $s(1)$ is a direct sum of atoms). There is a subset $K \subseteq J$ such that $c = \oplus_{i \in K} u_i$ and similarly $P \subseteq T$ such that $d = \oplus_{i \in P} u_i$. So $c \wedge d = 0$ follows from $a \wedge b = 0$ and hence $(\oplus_{i \in K} u_i) \wedge (\oplus_{i \in P} u_i) = 0$. Therefore $\{u_i\}_{i \in K \cup P}$ is an independent set in L (surely $K \cap P = \emptyset$). This independent set is contained in a maximal independent set of atoms in L ([2] lemma 8.4: one can use Zorn's Lemma in the set of all the independent subsets of L) say $1 = \oplus_{i \in R} u_i$ ([2] lemma 8.5) and $K \cup P \subseteq R \subseteq I$. We consider now $y = \oplus_{i \in R \setminus K} u_i$ so that $c \oplus y = 1$. From the modularity we deduce $c \oplus (y \wedge a) = a$ (using $c \leq a \leq c \oplus y = 1$) so if $y \wedge a \neq 0$ there is an atom $u_j \leq y \wedge a$ with $j \in I$ ([2] proposition 4.7) and even $j \in J$ (because $u_j \leq a$); hence $u_j \leq c$. But then $u_j \leq c \wedge (y \wedge a) = 0$, contradiction. So $y \wedge a = 0, c = a$ and $1 = a \oplus y$. We now continue simmetrically: let $x = \oplus_{i \in R \setminus P} u_i$ so that $d \oplus x = 1$. We see that $x \wedge b = 0$ and so $b = d \leq y$, because $P \subseteq R \setminus K$.$\Box$

Proposition 1.3 *Every modular, atomic, inductive lattice with enough complements is coatomic.*

Proof. Let $a \neq 1$ and b be a complement of a in L. Surely $b \neq 0$ (otherwise $a \vee b = 1 \Rightarrow a = 1$) so there is an atom $s \leq b$ such that $a \wedge s = 0$ (because $a \wedge b = 0$). L having enough complements, let m be a complement of s which contains a. In a modular lattice the complement of an atom is a maximal element and conversely, so that m is a maximal element of L.□

Consequence 1.1 *Every modular, inductive, atomic and complemented lattice is coatomic. -Every modular, inductive and reducible lattice is coatomic.*

In fact, we have the following

Theorem 1.1 *A modular inductive lattice L is reducible iff L is coatomic and every maximal element has a complement.*

Proof. A reducible, inductive lattice being complemented (cf.[2]) we need to justify only the converse of the second Consequence above. L being coatomic, if $s(L) = a \neq 1$, let m be maximal in $1/a$ and, by hypothesis, $c \neq 0$ a complement of m. As a complement of a maximal element c is an atom in L. But $a \leq m$ and $m \wedge c = 0$ implies $a \wedge c = 0$ or $s(L) \wedge c = 0$ so that $c \leq s(L)$ implies $c = 0$, contradiction.□

The lattice of an abelian group is modular and inductive. It is reducible iff the group is elementary.

Proposition 1.4 *Let$L$ be a modular lattice and $a \in L$. If $a \leq r(L)$ and $a/0$ is coatomic then a is superfluous in L.*

Proof. If a is not superfluous in L, there is $1 \neq c \in L$ such that $a \vee c = 1$. Then $a \leq c$ does not hold (otherwise $c = a \vee c = 1$) and hence $a \wedge c \neq a$. The sublattice $a/0$ being coatomic there is an element m maximal in $a/(a \wedge c)$. Using the modularity $m \vee c$ is maximal in $(a \vee c)/c = 1/c$. But in this case $a \leq r(L) \leq m \vee c$ and so $1 = a \vee c \leq m \vee c$, contradiction.□

In order to get the converse we must add some

Definition 1.3 *An element $d \in L$ is called* **divisible** *if $d/0$ contains no maximal elements. We say that L satisfies the condition (*) if in L every divisible element has a complement. L satisfies the condition (***D***) if for each $a \in L$ the sublattice $1/a$ satisfies* (*).

Remark 1.2 *In an algebraic (compactly generated) lattice one can easily prove (cf.[5]) that d is divisible iff $r(d/0) = d$or iff in $d/0$ all compact elements are superfluous.*

Proposition 1.5 *If a lattice L satisfies the condition* (**D**) *then for every superfluous element $a \in L$ the sublattice $a/0$ is coatomic and $a \leq r(L)$.*

Proof. The implication: a superfluous in $L \Rightarrow a \leq r(L)$, is well-known in every complete lattice. If $a/0$ is not coatomic, there is an element $b \neq a, b \leq a$ such that a/b has no maximal elements. Hence a is divisible in $1/b$ and has a complement c in $1/b$. Then $a \vee c = 1, a \wedge c = b$ and from $c \neq 1$ (otherwise $a \wedge c = b$ implies $a = b$) we deduce that a is not superfluous in L.□

In this way we have the following characterization

Theorem 1.2 *If the modular lattice L satisfies the condition* (**D**) *an element a is superfluous iff $a/0$ is coatomic and $a \leq r(L)$.*□

Remark 1.3 *In the lattice of all subgroups of an abelian group the condition* (**D**) *is clearly satisfied.*

- The characterization of the superfluous elements given above is somewhat dual to the following one(cf.[3]): in a modular algebraic lattice L which satisfies (**R3**) *an element a is essential iff $1/a$ is torsion and $s(L) \leq a$.*

- Dual to "in a complete atomic lattice the socle is the smallest essential element" one has also "in a complete coatomic lattice the radical is superfluous" and if the lattice is algebraic the radical is also the greatest superfluous element.

The relations between atomic and torsion lattices being recorded in [3] the next result connects coatomic lattices to reduced ones.

Proposition 1.6 *A modular, coatomic lattice which satisfies the condition* (**D**) *is reduced.*

Proof. If L is not reduced there is an element $0 \neq a \in L$ such that $a/0$ contains no maximal elements. But then a is divisible and let b be a complement of a in $L = 1/0$ (cf.(**D**)). Using modularity we have $a/0 = a/a \wedge b \cong a \vee b/b = 1/b$ so that $1/b$ has no maximal elements ($a \neq 0$ implies $b \neq 1$). Hence L is not coatomic.□

Proposition 1.7 *For a complete lattice L, $L \setminus \{1\}$ has a greatest element iff L is coatomic and $r(L)$ is maximal in L.*

Proof. If L has a greatest element $m \neq 1$ all the maximal elements coincide with m and with $r(L)$ so that every element $\neq 1$ is contained in m, so that L is coatomic.

Conversely, if $r(L)$ is maximal in L then $r(L)$ is the unique maximal element of L. If L is coatomic, every element is contained in $r(L)$ so that $1 \neq r(L)$ is the greatest element in $L \setminus \{1\}$.□

Remark 1.4 *Following [11] such lattices could be called* **local.**

- One easily gets: a lattice L is local iff $r(L)$ is superfluous and maximal.

2 Applications.

In what follows we characterize the class (dual to the cocyclic groups) of all the abelian groups G such that the lattice of all the subgroups of G has a greatest element $\neq G$ (is local) and the class of all the abelian groups G which have a unique maximal subgroup. For all the abelian group notions and notations we refer to [8].We first recall from [9]

Lemma 2.1 *G has a unique maximal subgroup iff there is a prime number p such that $|G/pG| = p$ and $qG = G$ for each prime $q \neq p$.*

A simple proof. The radical of the lattice of all subgroups of G is the Frattini subgroup of G, well-known as $\Phi(G) = \bigcap \{pG \mid p \text{ prime number } \}$ (cf.[7]). A subgroup M of G is maximal iff there is a prime number p such that G/M has p elements. Moreover $\Phi(G)$ is the intersection of all the maximal subgroups of G.

If M is the unique maximal subgroup of G then $\Phi(G) = M = pG$ for a prime number p, so that $|G/pG| = p$ and $qG = G$ for each prime number $q \neq p$ (otherwise $\Phi(G) \neq pG$). Conversely, from the hypothesis we deduce that pG is the unique maximal subgroup of G (we use the well-known result $pG = \bigcap \{M \mid M \text{ (maximal) subgroup of } G \text{ such that } |G/M| = p\}$).□

In what follows we call an abelian group **near-divisible** if it has a unique maximal subgroup , i.e. is q-divisible for every prime $q \neq p$ and $G/pG \cong \mathbb{Z}(p)$. Torsion near-divisible examples are $\mathbb{Z}(p^k)$ and $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p)$ and torsion-free near-divisible groups are $\mathbb{Q}_p$ and $\mathbf{J}_p$. Neither has coatomic subgroup lattice.

Proposition 2.1 *For a group G the lattice of all subgroups of G is local iff G is near-divisible and has no divisible factor groups.*

Proof. Using Lemma 3 and Proposition 6 the only argument left is: G has coatomic subgroup lattice iff G has no divisible factor groups. But this is obvious because a group is divisible iff it has no maximal subgroups and one has the well-known lattice isomorphism between an interval of subgroups of G, i.e.$[H, G)$ and the proper subgroups of G/H.$\square$

As mentioned above in what follows we determine the abelian groups with a local lattice of subgroups and the near-divisible groups.

The first task is simple. Indeed, groups with local lattice of subgroups are cyclic and indecomposable (see[11]: the so-called **hollow** groups are used). $\mathbb{Z}$ having not a local lattice of subgroups, the finite cocyclic groups (i.e.$\mathbb{Z}(p^k)$) are the only groups with local lattice of subgroups. Hence *the groups with a smallest proper subgroup are the cocyclic ones and the groups with a greatest proper subgroup are the finite cocyclic ones.*.

In order to determine the near-divisible groups we need some reductions.

Lemma 2.2 *G is near-divisible iff $G = D \oplus R$ for a non-zero near-divisible reduced group R and a divisible group D.*

Proof. Let D be the greatest divisible subgroup of G and $G = D \oplus R$ with R reduced. $R \neq 0$ because a divisible group is not near-divisible (it has no maximal subgroups). For each prime $q \neq p$ one has $D \oplus qR = qD \oplus qR = qG = G = D \oplus R$ so that $R = qR$. Moreover $D = \cap_t tD \subseteq \cap_t tG = pG$ where t denotes a prime number, so that $R/pR = R/R \cap pG \cong (R+pG)/pG = G/pG$. The converse is similar (and uses the same isomorphism).$\square$

Lemma 2.3 *Every near-divisible reduced group is indecomposable.*

Proof. Let G be a near-divisible reduced group and $G = A \oplus B$. For every prime $q \neq p$ we have $qA \oplus qB = qG = A \oplus B$ and hence $qA = A$, $qB = B$. Moreover, $G/pG \cong A/pA \oplus B/pB \cong \mathbb{Z}(p)$ so that for instance $|A/pA| = p$ and $B/pB = 0$. But hence B is divisible and $B = 0$.$\square$

Consequence 2.1 *Every near-divisible torsion reduced group is finite cocyclic.*$\square$

Indecomposable mixed groups do not exist so the problem reduces to the torsion-free case.

Lemma 2.4 *G is a near-divisible reduced torsion-free group iff G is isomorphic to a pure and dense subgroup of $\mathbf{J}_p$.*

Proof. In the above given hypothesis, the p-adic completion $\hat{G}$ of G is also near-divisible (because from $pG = p\hat{G} \cap G$ and $G + p\hat{G} = \hat{G}$ we deduce $G/pG \cong \hat{G}/p\hat{G}$). The direct summand B_0 of a basic subgroup B of $\hat{G}$ cannot decompose into more than one copy of $\mathbf{J}_p$ because $\mathbf{J}_p/p\mathbf{J}_p \cong \mathbb{Z}(p)$ and otherwise we would no more have $\hat{G}/p\hat{G} \cong \mathbb{Z}(p)$. So $B \cong \mathbf{J}_p$ and hence $\hat{G} \cong \mathbf{J}_p$.Conversely, each pure and dense subgroup A of $\mathbf{J}_p$ is near-divisible because $A + p\mathbf{J}_p = \mathbf{J}_p$ and $pA = A \cap p\mathbf{J}_p$ and hence $A/pA \cong \mathbf{J}_p/p\mathbf{J}_p \cong \mathbf{Z}(p)$.□

Finally we have

Theorem 2.1 *Each near-divisible group is a direct sum of a finite cocyclic or a pure dense subgroup of the p-adic integers with a possible zero divisible group.*□

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