MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE
DE L'APPROXIMATION

MATHEMATICA, Tome 23 (46), Nº 1, pp. 9—10

NOTE ON B-HIGH SUBGROUPS OF ABELIAN GROUPS

by

GRIGORE CĂLUGĂREANU

In [1], FUCHS proved the following simple result:

L e m m a — *Let* $B \leqslant A$ *and let* C *be* B*-high. Then* $A^* = B \oplus C$ *satisfies*: (a) A/A^* *is a torsion group*; (b) $(A/A^*)\ [p] \cong (pA+C)\cap B/pB$ *for every prime* p.

In what follows we generalise this result and prove the converse. For this purpose we make an immediate application of the „snake" lemma, well-known result of homological algebra. All the groups are abelian.

Let B and C be subgroups of the group A. The canonic homomorphism $B \to A \to A/C$ is easily embedded in the exact sequence $0 \to B \cap C \to \to B \to A/C \to A/(B+C) \to 0$.

Let f be an endomorphism of A such that $f(B) \leqslant B$, $f(C) \leqslant C$ and $B \cap C = 0$. One can naturally extend the above exact seuqence to the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & A/C & \to & A/(B\oplus C) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & (f(A)+C)\cap B & \to & (f(A)+C)/C & \to & G & \longrightarrow & 0 \end{array}$$

where $G = (f(A)+C)/(f(A)+C) \cap (B \oplus C) \cong (f(A)+C)+(B\oplus C)/(B\oplus C)$ and the vertical homomorphisms are trivially induced by f.
This last diagram is appropriate to the application of the „snake" lemma. Thus, we obtain the following exact sequence:

$$B \cap \operatorname{Ker}(f) \to f^{-1}(C)/C \xrightarrow{\alpha_f} f^{-1}(B\oplus C)/(B\oplus C) \xrightarrow{\delta_f} (f(A)+C)\cap B/f(B) \to 0$$

Here $\alpha_f(a+C) = a+(B\oplus C)$ and $\delta_f(a+(B\oplus C)) = \pi_B(f(a))+f(B)$ where $\pi_B : B \oplus C \to B$ is the canonic projection from the direct sum. Hence, we can state our generalisation as follows

PROPOSITION — *Let* B *and* C *be disjoint subgroups of the group* A. *If* f *is an endomorphism of* A *such that* $f(B) \leqslant B$ *and* $f(C) \leqslant C$ *then an epimorphism* $\delta_f : f^{-1}(B\oplus C)/(B\oplus C) \to (f(A)+C) \cap B/f(B)$ *always exists. Moreover,* δ_f *is isomorphism iff* $f^{-1}(C)/C \leqslant (B\oplus C)/C$.

The last assertion follows simply using the exactness of the „connecting" sequence. Indeed, δ_f is isomorphism iff Ker $\delta_f = \text{im } \alpha_f = 0$, and one easily checks that im $\alpha_f = 0$ is equivalent with the stated condition. ■

Remark. — If $f^{-1}(C)/C \leqslant S(A/C)$ and C is B-high then δ_f is isomorphism. Indeed, this follows immediately the following three conditions being equivalent:

(i) C is B-high; (ii) $B \oplus C/C$ is essential in A/C;

(iii) $A/B \oplus C$ is torsion and $S(A/C) \leqslant B \oplus C/C$.

A special case is now obtained by taking f to be the multiplication by a positive integer m. The epimorphism $\delta_m : (A/B \oplus C)[m] \to (mA + C) \cap \cap B/mB$ is an isomorphism iff $(A/C)[m] \leqslant B \oplus C/C$. Hence, if m is square-free and C is B-high then δ_m is isomorphism.

Now we are able to prove the converse of Fuchs's lemma

PROPOSITION. — *Let B, C be disjoint subgroups of A. If $A/B \oplus C$ is torsion and all the epimorphisms δ_p, for every prime p, are isomorphisms then C is B-high.*

One has only to use $S(A/C) = \oplus_p(A/C)[p]$ and the condition (iii) mentioned above. ■

The „connecting" epimorphisms δ_p are also present in GRÄTZER lemma [2, pp. 49] which is in this way straightforward

Lemma. — *Let C be B-high in A. $A = B \oplus C$ iff for every prime p, δ_p is trivial.* ■

Hence, B is an absolute direct summand iff for every B-high subgroup C, all the δ_p are trivial.

REFERENCES

[1] Fuchs, L., *On a useful lemma for abelian groups*, Acta Sci. Math., (Szeged), 17 (1956) 134—138.

[2] Fuchs, L., *Infinite Abelian Groups*, vol. I, New York 1970.

Received, 29. IX. 1980