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# ABELIAN GROUPS WITH CONTINUOUS LATTICE OF SUBGROUPS

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ABSTRACT. — If $A$ is an abelian group, the lattice of all the subgroups of $A$ is lower continuous iff $A$ is a torsion group with finitely cogenerated $p$-components.

A complete lattice $L$ is called lower continuous if $a \vee (\bigwedge_{c \in C} c) = \bigwedge_{c \in C} (a \vee c)$ holds for every element $a \in L$ and every chain $C$ in $L$. If $A$ is an abelian group we denote by $L(A)$ the lattice of all the subgroups of $A$. This is a modular, compactly generated and hence upper continuous lattice. Our result is the following

THEOREM *$L(A)$ is lower continuous iff $A$ is a torsion group with finitely cogenerated $p$-components (i.e. direct sums of cocyclic $p$-groups).*

This result does not exceed very much the following general one

1. PROPOSITION. *Every complete artinian lattice is lower continuous. Indeed, if a poset $P$ satisfies the descending chain condition, every chain $C$ from $P$ has a least element and the above condition is obvious.*

2. COROLLARY. *If $A$ is a finitely cogenerated abelian group then $L(A)$ is lower continuous.*

This is immediate using a well-known characterization [1,25.1]: $L(A)$ is artinian iff $A$ is finitely cogenerated.

Our theorem will follow from a few lemmas.

3. LEMMA. *If $B$ is a subgroup of $A$ and $L(A)$ is lower continuous then $L(B)$ is lower continuous too.*

4. LEMMA *$L(\mathbf{Z})$ is not lower continuous.*

*Proof.* Let $p$ and $q$ be different primes, $C = \{p^n\mathbf{Z}\}_{n \in N}$ the descending chain of subgroups and $B = q\mathbf{Z}$. Then $B + p^{\omega}\mathbf{Z} = B \neq \mathbf{Z} = \bigcap_{n \in N} (B + p^n\mathbf{Z})$ because $p^{\omega}\mathbf{Z} = 0$ and $q\mathbf{Z} + p^n\mathbf{Z} = \mathbf{Z}$.

5. LEMMA. *If $A$ is an infinite elementary group, $L(A)$ is not lower continuous.*

*Proof.* Using well-known reduction theorems we can suppose $A$ to be a countable direct sum of $\mathbf{Z}(p)$ for a prime $p$. Let $\{e_n = (0, \ldots, 0, 1, 0, \ldots) / n \in N^*\}$ the canonic basis of $A$ (as linear space over $(\mathbf{Z}p)$), $B$ the subgroup generated by $\{e_n / n \geqslant 2\}$ and $C_n$ the subgroups generated by $\{v_k / k \geqslant n\}$

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where $v_k = \sum_{s=1}^{k} e_s$ ($n \in N^*$). The following relations hold: $A = C_1 \supset C_2 \supset \ldots \supset \ldots \supset C_n \supset \ldots$, $\bigcap_{n \in N^*} C_n = 0$, $e_1 \notin B$ and $e_1 = (p-1) \sum_{s=2}^{n} e_s + v_n \in B + C_n$ for each $n \in N^*$. Hence $A$ is as stated. This example was suggested by Zoltan Finta.

*Proof of the theorem*. Let $A$ be an abelian group and $L(A)$ a lower continuous lattice. From 3 and 4, $A$ is a torsion group. If $p$ is a prime let $A_p$ be the $p$-component of $A$ and $A[p]$ its socle. Using 5, $A[p]$ is finite (otherwise it would contain a countable (elementary) subsocle) and hence $A_p$ is finitely cogenerated [1, 25.1].

Conversely, let $A$ be a torsion group such that $A_p$ is finitely cogenerated for every prime $p$, $\{C_i\}_{i \in I}$ a chain of subgroups of $A$ and $B$ a subgroup of $A$. The inclusion $\bigcap_{i \in I} (B + C_i) \subseteq B + (\bigcap_{i \in I} C_i)$ is readily verified going down to $p$-components, using 2 and the obvious equality $(\bigcap_{i \in I} (X_i)_p) = (\bigcap_{i \in I} X_i)_p$.

## BIBLIOGRAPHY

1. Fuchs L., *Infinite abelian groups*, vol. 1, Academic Press, New York, London, 1970.