To my father, as a token of gratitude

A VARIETY OF ASSOCIATIVE STRUCTURE WITH ONE-SIDED ZERO ELEMENTS IN AUTONOMOUS CATEGORIES

BY

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Using the "lifting" of a well-known bijection in *Ens* and the definition of the multiplicative systems in an autonomous category, a variety in the Frölich sense is found. The variety contains multiplicative systems "lifted" over associative structures of one-sided zero elements.

1. *Introduction*. It is a well known fact that in the category *Ens*, the following bijection holds

$$\alpha_{AB} : \text{Hom}(A, B) \cong \mathop{\times}_{a \in A} B$$

where A, B are arbitrary sets and $\alpha_{AB}(f) = \{f(a)\}_{a \in A}$.

In categories with richer structure, like Ab or Mod_R, α remains only an injective homomorphism, i.e. a monomorphism in these categories.

Let $\mathcal{a}$ be a closed category in the sense of (2). We shall denote by (A, B) the values of the lifted hom-functor $\text{hom}_{\mathcal{a}} : \mathcal{a}^{op} \times \mathcal{a} \to \mathcal{a}$, and by $|A|$ the value of the set-valued functor of subjacency $|-| : \mathcal{a} \to Ens$, on objects A, B in $\mathcal{a}$.

In (4), if A, B are objects in $\mathcal{a}$ and $a \in |A|$, the "evaluation in a"-morphism is constructed as follows

$$ev_a : (A, B) \to B$$

is the image of a by the subjacent map of the $\mathcal{a}$-morphism

$$A \to ((A, B), B)$$

corresponding to the identity of (A, B) in the bijection

$$|((A, B), (A, B))| \cong |(A, ((A, B), B))|.$$

One can easily show that for every $f \in |(A, B)|$ we have

$$|ev_\alpha|(f) = |f|(\alpha).$$

If $\mathcal{A}$ has products, the family $\{ev_\alpha : (A, B) \to A\}_{\alpha \in |A|}$ induces by factorization a morphism $\alpha_{AB} : (A, B) \to \prod_{\alpha \in |A|} B$.

Assuming that the functor of subjacency is faithful and preserves products (assumptions that hold in most of the concrete categories) we can trivially show that α is monic and we have

$$|\alpha|(f) = \{|f|(\alpha)\}_{\alpha \in |A|}.$$

The commutativity of the two following diagrams shows that α_{AB} is natural in his both arguments :

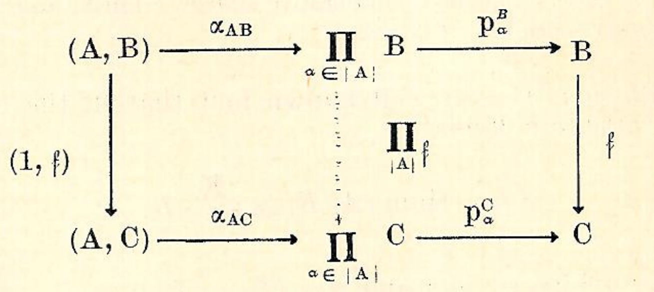

Diagram 1

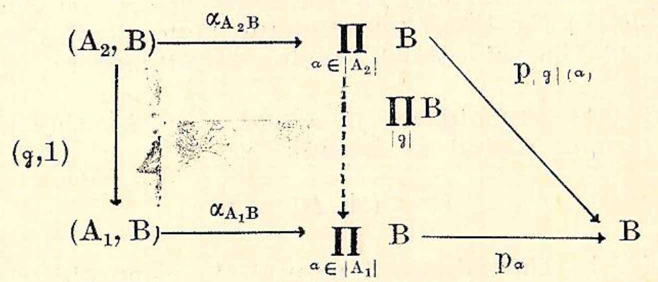

Diagram 2

The commutativity holds because equalities like

$$ev_\alpha^C \cdot (1, f) = f \cdot ev_\alpha^B, \quad ev_\alpha \cdot (g, 1) = ev_{|g|(\alpha)}$$

for any $f : B \to C$ and $g : A_1 \to A_2$ are easily established at the subjacent level (this is sufficient because of the faithfulness of $|-|$).

2. Definitions. Let $\mathcal{V}$ be a closed category. We define, as in (1) the category $\mathcal{M}$ of *multiplicative systems* in $\mathcal{V}$; that is, the objects of $\mathcal{M}$ are pairs (x, π) where $x \in \mathcal{V}$ and $\pi : x \to (x, x)$ is an arrow of v, and the morphisms are the arrows $f : (x, \pi) \to (x', \pi')$ which are just arrows $f : x \to x'$ in $\mathcal{V}$ such that the following pentagon commutes.

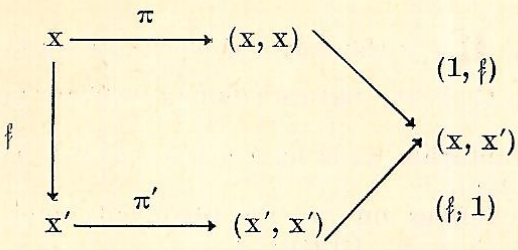

Diagram 3

It is easy to see that every multiplicative system defines an operation on $|x|$ in a trivial way : for $a, b \in |x|$, $a \circ b = \| \pi | (a) | (b)$.

First, we shall show that *if $\mathcal{V}$ is a symmetric monoidal closed category with products, then $\mathcal{M}$ has products.*

Let $\{(x_i, \pi_i)\}_{i \in I}$ be a family of multiplicative systems. Denoting by P the product in $\mathcal{V}$ $\prod\limits_i x_i$, we are looking for a morphism, $\pi : P \to$ $\to (P, P)$ induced by the family $\{\pi_i\}$.

We consider the family

$$\left\{ P \xrightarrow{pr_i} x_i \xrightarrow{\pi_i} (x_i, x_i) \xrightarrow{(pr_i, 1)} (P, x_i) \right\}_{i \in I} .$$

$\mathcal{V}$ being symmetric monoidal closed, the functor $\mathcal{V}(P, -) : \mathcal{V} \to \mathcal{V}$ has a left (strong) adjoint namely $- \otimes P : \mathcal{V} \to \mathcal{V}$, and therefore preserve limits and in particular, products.

It follows that $\prod\limits_{i \in I} (P, x_i) \cong (P, \prod\limits_i x_i) = (P, P)$, and so the above family gives by factorization the morphism $\pi : P \to (P, P)$ we have been looking for. In this way, $pr_i : (P, \pi) \to (x_i, \pi_i)$ is a morphism in $\mathcal{M}$.

Remark. The multiplicative system (P, π) is canonic enough; in fact, one can show that the "operation" π is induced by $\{\pi_i\}$ "componentwise".

Recall from (3) that a *variety* $\mathcal{B}$ in a category $\mathcal{C}$ is a full subcategory of $\mathcal{C}$, satisfying the following axioms :

(i) If $f : A \to B$ is an epimorphism and $A \in \mathcal{B}$ then $B \in \mathcal{B}$

(ii) If $f : A \to B$ is an monomorphism and $B \in \mathcal{B}$ then $A \in \mathcal{B}$

(iii) If $(A_i)_{i \in I}$ is an indexed set of objects, $(A_i) \subset \mathcal{B}$ then

$$\prod A_i \in \mathcal{B}.$$

3. *The variety.* Let $\widetilde{\mathcal{M}}$ be the full subcategory of $\mathcal{M}$ whose objects are multiplicative systems (x, π_x) where π_x is such, that the two morphisms

$$x \xrightarrow[\pi_x]{} (x, x) \xrightarrow[\alpha_{xx}]{} \prod_{|x|} x, \qquad x \xrightarrow{\Delta_x} \prod_{|x|} x$$

are equal. $\Delta_x : x \to \prod_{|x|} x$ is the *diagonal morphism* induced by the family $\{id_x : x \to x\}_{|x|}$ and α_{xx} is the natural monomorphism defined in the first section.

Remark. The notation π_x is legitimate : if such a morphism exists, it is unique (α monic).

Remark. $|\pi_x|$ defines on $|x|$ an operation such that for every $i, j \in |x|$ we have $i \circ j = \| \pi_x | (i) | (j) = i$.

Thus, all the elements of $|x|$ are all left-zero or all right-zero elements for the operation $|\pi_x|$. It is easy to see that this operation is associative, but not commutative.

THEOREM. *Let $\mathcal{M}$ be the category of multiplicative systems on an autonomous category $\mathfrak{a}$ whose subjacent functor preserves products. The full subcategory $\widetilde{\mathcal{M}}$ is a variety in $\mathcal{M}$.*

Proof. (i) Let $f : (x, \pi_x) \to (x', \pi')$ be an epimorphism in $\mathcal{M}$. Consider the diagram

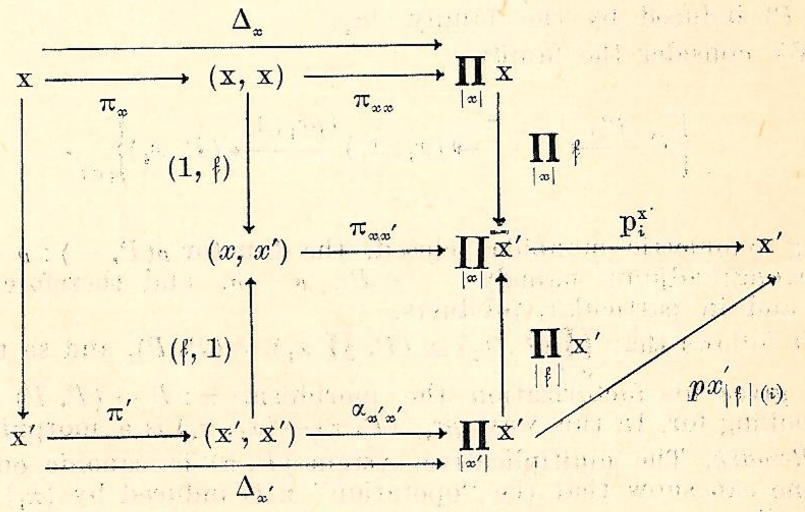

Diagram 4

where $i \in |x|$.

Since α is natural, and f is a morphism in $\mathcal{M}$, the pentagon, the three squares and the triangle are commutative. The two top lines are equal by the definition of π_x. Hence, the bottom lines are also equal since f is surjective (and so $\{p^{x'}_{|f|(i)}\}_{i \in |x|}$ are *all* the projections from $\prod_{|x'|} x'$) and f is an epimorphism, and we have $\Delta_{x'} = \alpha_{x'x'} \cdot \pi'$. Thus $\pi' = \pi_{x'}$ and $(x', \pi') \in \widetilde{\mathcal{M}}$.

(ii) Now, let $\mathfrak{f} : (x, \pi) \to (x', \pi_{x'})$ be a monomorphism in $\mathcal{M}$. Consider the similar diagram

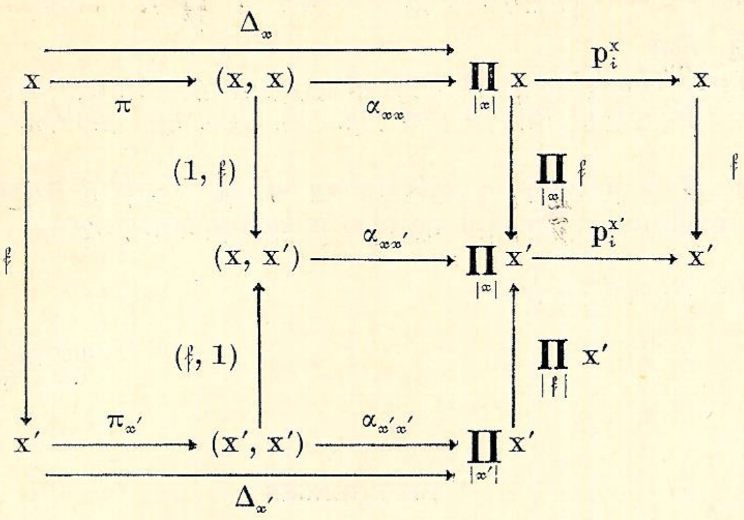

Diagram 5

where $i \in |x|$.

The subdiagrams are commutative by similar arguments. The bottom lines are equal by the definition of $\pi_{x'}$. Hence the top lines are equal, i.e. $\Delta_x = \alpha_{xx}\pi$, since $\mathfrak{f}$ is monic and $\{p_i^x\}_{i \in |x|}$ are *all* the projections from the product $\prod_{|x|} x$. Thus $\pi = \pi_x$ and $(x, \pi) \in \widetilde{\mathcal{M}}$.

(iii) We finally consider the diagram

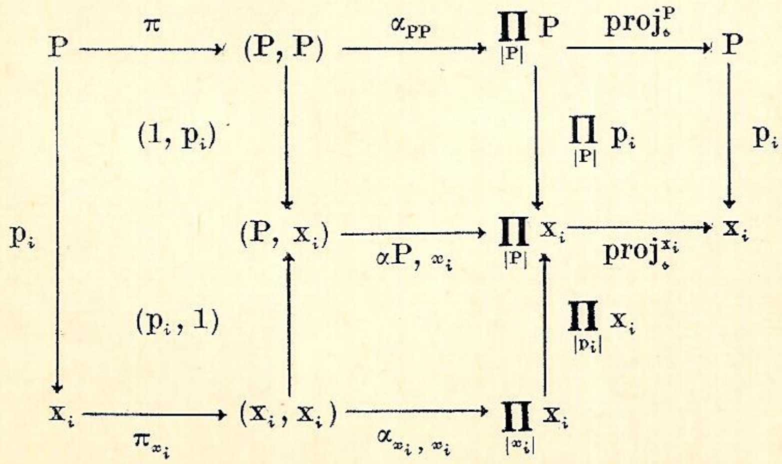

Diagram 6

where $(x_i, \pi_{x_i}) \in \mathcal{M}$, P has the projections p_i and $\mathfrak{s} \in |P|$. The top lines are equal and as the subdiagrams are again commutative, the bottom lines are equal, i.e. $\Delta_P = \alpha_{PP}\pi$, since $\{p_i\}_{i \in I}$ are *all* the projections of P and $\{\mathrm{proj}^P_{\mathfrak{s}}\}$, $\mathfrak{s} \in |P|$ are *all* the projections of $\prod_{|P|} P$. Thus $\pi = \pi_P$ and $(P, \pi) \in \widetilde{\mathcal{M}}$, qed.

Remark. If ϑ *is complete, then* $\mathcal{M}$ *is* easily seen to be *complete* because $\vartheta \varprojlim_{i \in I} x_i, -)$ preserves limits) *and* one can give an analogous proof to show that $\widetilde{\mathcal{M}}$ *is a complete subcategory of* $\mathcal{M}$. Without considering the injective anvelopes, $\widetilde{\mathcal{M}}$ could be also a *monosubcategory* (5).

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REFERENCES

1. DUBUC EDUARDO, STREET ROSS, *Dinatural Transformations*, Reports on the Midwest Category Seminar IV, Lecture Notes 137, p. 126–137, Springer–Verlag, 1970.
2. EILENBERG SAMUEL, KELLY G. MAX, *Closed Categories*, Proc. Conf. Categorical Algebra, La Jolla, 1965, p. 421–562, Springer-Verlag Berlin, Heidelberg, New York, 1966.
3. FRÖLICH A., *Baer-invariants of algebras*, Trans. Amer. Math. Soc., 1963, **109**, 221–224.
4. LINTON F. E. J., *Autonomous categories and duality of functors*, Journal of Algebra, 1965, **2**, 315–349.
5. MITCHELL BARRY, *Theory of categories*, Academic Press, New York, London, 1965.