A 2×3 association analogue

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Proposition 1 *Let $a, b, c, d \in R$, a GCD (commutative) domain, such that $ad = bc$. If $\delta = \gcd(a, b)$ and $\lambda = \gcd(c, d)$ and $a = \delta a_1$, $b = \delta b_1$, $c = \lambda c_1$, $d = \lambda d_1$ then a_1, c_1 and b_1, d_1 are associated in divisibility, respectively. Moreover, if $c_1 = a_1 u$ with $u \in U(R)$ then $d_1 = b_1 u$, for the same unit u.*

Proof. By cancellation with $\delta\lambda$ we have $a_1 d_1 = b_1 c_1$. Since a_1, b_1 are coprime we get $a_1 \mid c_1$. Since c_1, d_1 are coprime, we also obtain $c_1 \mid a_1$, as desired. Analgous for b_1, d_1.

Assume $c_1 = a_1 u$ (and so $a_1 = c_1 u^{-1}$), and $b_1 = d_1 v$ (and so $d_1 = b_1 v^{-1}$). Since $a_1 d_1 = b_1 c_1$, we get $a_1 d_1 = a_1 d_1 uv$ so $v = u^{-1}$ follows from $uv = 1$. Hence $d_1 = b_1 v^{-1} = b_1 u$. ■

Remark. The hypothesis corresponds to $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$, that is, dependent rows: $d \begin{bmatrix} a & b \end{bmatrix} = b \begin{bmatrix} c & d \end{bmatrix}$. Dividing by their gcd we get a_1, c_1 and b_1, d_1 are associated in divisibility, respectively. Moreover, $\begin{bmatrix} a_1 & b_1 \end{bmatrix} u = \begin{bmatrix} c_1 & d_1 \end{bmatrix}$.

THE QUESTION: Is the 2×3 analogue true or false ? Yes, TRUE.
That is

Conjecture 2 *If $\operatorname{rk} \begin{bmatrix} a & b & c \\ a' & b' & c' \end{bmatrix} = 1$ (i.e., $ab' = a'b$, $ac' = a'c$, $bc' = b'c$), $\delta = \gcd(a, b, c)$, $\lambda = \gcd(a', b', c')$ and $a = \delta a_1$, $b = \delta b_1$, $c = \delta c_1$, $a' = \lambda a'_1$, $b' = \lambda b'_1$ and $c' = \lambda c'_1$, then then a_1, b_1, c_1 and a'_1, b'_1, c'_1 are respectively associated (in divisibility). Moreover, $\begin{bmatrix} a'_1 & b'_1 & c'_1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} u$ for some $u \in U(R)$.*

The *proof of the second claim* is easy: now $ab' = a'b$ becomes $\delta a_1 \lambda b'_1 = \lambda a'_1 \delta b_1$ and so $a_1 b'_1 = a'_1 b_1$. Then, by association, if $a'_1 = a_1 u$, $b'_1 = b_1 v$, we get $a_1 b_1 v = a_1 u b_1$ whence $u = v$. If also $c'_1 = w c_1$ we obtain $u = v = w$.

Proof for the first claim. We just use the following

Lemma 3 *Let $a, b, c, a', b', c' \in R$, a GCD (commutative) domain. If $ab' = a'b$, $ac' = a'c$, $bc' = b'c$ and the rows $\begin{bmatrix} a & b & c \end{bmatrix}$ and $\begin{bmatrix} a' & b' & c' \end{bmatrix}$ are unimodular then the pairs a, a', b, b' and c, c' are associated. Moreover, there exists a unit $u \in U(R)$ such that $\begin{bmatrix} a' & b' & c' \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix} u$.*

Proof. Denote $\delta = \gcd(a, b)$ with $a = \delta a_1$, $b = \delta b_1$ and $\delta' = \gcd(a', b')$ and $a' = \delta' a'_1$, $b' = \delta' b'_1$. From $ab' = a'b$ cancelling $\delta\delta'$ we obtain $a_1 b'_1 = a'_1 b_1$. Since a_1, b_1 are coprime, it follows $a_1 \mid a'_1$. Symetrically, since a'_1, b'_1 are coprime, it follows $a'_1 \mid a_1$, so that a_1, a'_1 are associates. Hence there is a unit $u \in U(R)$ such that $a_1 = a'_1 u$.

Further, notice that $\gcd(\delta, c) = \gcd(\gcd(a, b), c) = 1$ and so δ, c are coprime. Now we use $ac' = a'c$, that is, $\delta(a'_1 u)c' = \delta a_1 c' = \delta' a'_1 c$. Cancelling a'_1 we get $\delta u c' = \delta' c$ and since δ, c are coprime, $\delta \mid \delta'$. Symmetrically, $\delta' \mid \delta$ and so δ, δ' are also associates. Therefore $a = \delta a_1$ and $a' = \delta' a'_1$ are associates.

In a similar way, it follows that b, b' and c, c' are associates, respectivelly.

Finally, suppose $a' = ua$, $b' = vb$ and $c' = wc$ for some $u, v, w \in U(R)$. From $ab' = a'b$ we get $avb = uab$, so $v = u$. Analogously, $w = v$ and so $w = v = u$, as claimed. ■