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# UU rings

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ABSTRACT. A new class of rings is studied: rings all whose units are sums $1 + n$, for a suitable nilpotent element $n$. These are called UU rings.

## 1. INTRODUCTION

A ring was called *clean* if every element is a sum of a unit and an idempotent, *nil-clean* if every element is a sum of a nilpotent and an idempotent (adding "strongly" when these elements commute), and $\pi$*-regular* if for every $a \in R$ the descending chain $aR \supseteq a^2R \supseteq ...$ (or equivalently $Ra \supseteq Ra^2 \supseteq ...$ ) stabilizes (or, if and only if there is $n \geq 1$ such that $a^n \in a^{n+1}R \cap Ra^{n+1}$).

While nil-clean rings are clean (see [2] or [3]), the converse fails, but holds if all units are unipotent (i.e., $1 + n$ for a nilpotent element $n$). Moreover, strongly nil-clean rings are strongly $\pi$-regular, the converse fails, but holds if and only if all units are unipotent (see [3], Corollary 3.11).

These two facts suggest that the class of rings all whose units are unipotent could be studied independently.

In this paper, this is what we do. Such rings are called UU (units are unipotent). As mentioned above, strongly nil-clean rings are UU.

We show that this property passes to corners, is not Morita invariant and is not closed under extensions. UU rings have nil radical and matrix rings have not this property. Finally we characterize the Abelian groups whose endomorphism rings share this property.

## 2. BASIC PROPERTIES AND EXAMPLES

In the sequel, $R$ denotes a nonzero ring with identity, $N(R)$ its nilpotent elements and $U(R)$ the units of $R$. The ring $R$ will be called a *domain* if it has no nonzero zero divisors (not necessarily commutative).

It is well-known that $1 + n$ *is a unit, whenever* $n$ *is nilpotent*, that is $1 + N(R) \subseteq U(R)$. The converse fails (e.g., in $\mathbf{Z}(3)$, $\overline{2}$ is a unit but $\overline{1} = \overline{2} - \overline{1}$ is not nilpotent).

To make the difference, an element $a \in R$ in a ring is (sometimes) called *unipotent*, if $a - 1$ is nilpotent. That is, the units in $1 + N(R)$ are called *unipotent*.

We can always consider an injective function $f : N(R) \longrightarrow U(R)$, $f(n) = 1+n$, which is a bijection between nilpotent and unipotent elements. Also notice that $N(R) \cap U(R) = \varnothing$.

**Definition 2.1.** A ring is called *UU* if all **u**nits are **u**nipotent (that is, $U(R) \subseteq 1 + N(R)$, and so $1 + N(R) = U(R)$). In this case, the above bijection is between nilpotent and units, and so $|N(R)| = |U(R)|$, these sets have equal cardinals. Notice that (only) if these are

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finite cardinals, this condition is also sufficient for a ring to be UU (the injective function $f$ is also surjective).

If $R \setminus U(R) \subseteq N(R)$ then $R = (1 + N(R)) \cup N(R)$. In the finite case, if this happens, half of elements are nilpotent, half are units (i.e., unipotents).

**Remark 2.1.** 1) Since for any reduced ring $R$, $N(R) = \{0\}$, in a reduced UU ring, $1$ is the only unit. Since clearly $-1$ is also a unit, $-1 = 1$ and $2 = 0$. This way, $\mathrm{char}(R) = 2$. As a special case, the ring of all integers, $\mathbf{Z}$ is not UU.

2) $R \setminus U(R) \subseteq N(R)$ generally fails for UU rings (i.e., there exist elements which are not units nor nilpotents). An example is $\mathrm{End}(\mathbf{Z}(2) \oplus \mathbf{Z}(4))$ (see Section 3).

3) Since $-1$ is a unit, in a UU ring $2$ (or $-2$) is nilpotent. Therefore, domains (or fields) with at least 3 elements are not UU. Moreover, modulo nilpotent elements, such a ring must have characteristic $2$.

Recall that an element $r \in R$ is said to be *quasiregular*, if $1 - r$ is a unit in $R$. It is well-known that every nilpotent element is quasiregular. Since $1 - r \in U(R)$ is equivalent to $r - 1 \in U(R)$, the set $1 + U(R)$ is exactly the set of all quasiregular elements. It is readily seen that a ring $R$ is UU if and only if every quasiregular element is nilpotent.

**Proposition 2.1.** *A ring $\mathbf{Z}(m)$ is UU if and only if $m = 2^k$ for a suitable positive integer $k$.*

*Proof.* Suppose $m = p_1^{k_1} ... p_s^{k_s}$. Then $m - 1 \in U(\mathbf{Z}(m))$ and $m - 2 \in N(\mathbf{Z}(m))$ if and only if $p_1 ... p_s | m - 2$. But this happens only if $p_1 ... p_s | 2$, i.e., $m = 2^k$ for a suitable $k$. Conversely, $\mathbf{Z}(2^k)$ is UU. Indeed, in this (commutative) ring, the even classes are nilpotent and the odd classes are units (and so unipotents). □

**Corollary 2.1.** *The implication: $R/I$ is UU $\Longrightarrow$ $R$ is UU, generally fails.*

Indeed, $\mathbf{Z}/2\mathbf{Z}$ is UU but $\mathbf{Z}$ is not.
However

**Proposition 2.2.** *Let $I$ be a nil ideal in a ring $R$ with $N(R)$ closed under addition. If $R/I$ is UU then $R$ is UU.*

*Proof.* Suppose $u \in U(R)$. Then $u + I \in U(R/I)$ and by hypothesis, $u + I = 1 + n + I$ with $n + I$ nilpotent in $R/I$. Since $I$ is a nil ideal, $n$ is also nilpotent in $R$ and so there is $n' \in I$, and so nilpotent in $R$, with $u = 1 + n + n'$. Finally, $n + n' \in N(R)$, as desired. □

**Remark 2.2.** There is *no relationship between UU rings and local rings*. Indeed, $\mathbf{Z}(3)$ is local but not UU, and any Boolean ring with $> 2$ elements is UU but not local. We come back to this below.

Recall the following:

**Proposition 2.3.** *Let $R$ be a commutative ring with identity, and $f = a_0 + a_1 X + ... + a_n X^n \in R[X]$ be a polynomial. Then*

*a) $f$ is a unit in $R[X]$ if and only if $a_0$ is a unit in $R$ and $a_1, a_2, ..., a_n \in N(R)$ are nilpotent in $R$;*

*b) $f$ is nilpotent in $R[X]$ if and only if all the coefficients are nilpotent.*

Therefore

**Corollary 2.2.** *A polynomial ring $R[X]$ over a commutative ring with identity is UU if and only if $R$ is UU.*

*Proof.* Indeed, for any nilpotent polynomial $f$, $1+f=(1+a_0)+a_1X+...+a_nX^n$. Every unit is unipotent in $R[X]$ if and only if $R$ has this property. $\square$

**More examples.** 1) $(\mathbf{Z}/2^n\mathbf{Z})[x]$. Or, more generally, rings which, modulo a nil ideal, have characteristic 2 and no units but 1.

2) Local rings with nilpotent maximal ideals and residue field $\mathbf{Z}/2\mathbf{Z}$.

**Proposition 2.4.** *Every nil-clean ring is clean. The converse is true for UU rings.*

*Proof.* Let $a \in R$ be arbitrary. Then $a-1=e+n$ and so $a=e+(1+n)=e+u$. Conversely, let $a \in R$ be arbitrary. Then $a+1=e+u$ and so $a=e+(u-1)=e+n$ since $u \in U(R)$ is unipotent (by hypothesis). $\square$

**Proposition 2.5.** *UU passes to corners.*

*Proof.* Let $e \in R$ be an idempotent. First notice that if $u \in U(eRe)$ then $u-\overline{e} \in U(R)$. Further, if $R$ is a UU ring, $u-\overline{e}-1 \in N(R)$, that is $(u-\overline{e}-1)^k=0$ for some positive integer $k$. Finally, the following computation shows that $(u-e)^k=0$: multiply from left $(u-\overline{e}-1)^k=0$ by $e$. Then $(u-e)(u-\overline{e}-1)^{k-1}=0$ and since $u-e=(u-e)e$ we obtain $(u-e)^2(u-\overline{e}-1)^{k-2}=0$. After similar $k-2$ steps, $(u-e)^k=0$ follows and so $u-e \in N(eRe)$, as desired. $\square$

**Remark 2.3.** The above computation may be avoided if we notice that, since $R$ is UU, $u-\overline{e} \in U(R)$ implies $1+u-(1-e)=u+e$ is nilpotent. Hence $eRe$ is UU.

**Proposition 2.6.** *UU rings have nil radical. The converse is not true.*

*Proof.* Suppose $R$ is UU. Then $J(R) \subseteq 1+U(R)=N(R)$, so the radical is nil. Conversely, as seen before, $\mathbf{Z}(3)$ is a counterexample. $\square$

This generalizes half of the characterization of strongly nil-clean rings in [2] (p. 51: for a ring $R$ with only trivial idempotents, it is proved that $R$ is nil clean if and only if $R$ is a local ring with $J(R)$ nil and $R/J(R)=F_2$. Such a ring is further strongly nil clean.)

Finally, observe that $\mathrm{End}(\mathbf{Z}(2)\oplus\mathbf{Z}(2)) \cong \mathcal{M}_2(F_2)$ is not UU. Indeed, (use Corollary 3.3, or) $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is a unit which is not unipotent.

Hence, *split extensions of UU rings need not be UU*. This example also shows that *UU is not Morita invariant* (i.e., $\mathcal{M}_2(F_2)$ is not UU, even though $F_2$ is).

## 3. UU MATRIX RINGS

In this short section, we show that no matrix rings are UU. First we give an elementary proof.

**Proposition 3.7.** *For any $n>1$ and any non-zero ring $R$, there exist invertible matrices $U \in \mathcal{M}_n(R)$ such that $U+I_n$ is also invertible.*

*Proof.* In the general odd case take the unit

$$U_{2k+1}=\begin{bmatrix} -1 & 0 & 0 & & & & & & 0 & 1 & 1 \\ 0 & -1 & 0 & & & & & & 1 & 1 & 0 \\ 0 & 0 & -1 & & & & & & 1 & 0 & 0 \\ & & & \ddots & & & & \vdots & & & \\ & & & & -1 & 1 & 1 & & & & \\ & & & & 1 & -1 & 0 & & & & \\ & & & & 1 & 0 & 0 & & & & \\ & & & \vdots & & & & \ddots & & & \\ 0 & 0 & 1 & & & & & & 0 & 0 & 0 \\ 0 & 1 & 0 & & & & & & 0 & 0 & 0 \\ 1 & 0 & 0 & & & & & & 0 & 0 & 0 \end{bmatrix}, \text{ for which}$$

$$U_{2k+1}+I_{2k+1}=\begin{bmatrix} 0 & 0 & 0 & & & & & & 0 & 1 & 1 \\ 0 & 0 & 0 & & & & & & 1 & 1 & 0 \\ 0 & 0 & 0 & & & & & & 1 & 0 & 0 \\ & & & \ddots & & & & \vdots & & & \\ & & & & 0 & 1 & 1 & & & & \\ & & & & 1 & 0 & 0 & & & & \\ & & & & 1 & 0 & 1 & & & & \\ & & & \vdots & & & & \ddots & & & \\ 0 & 0 & 1 & & & & & & 1 & 0 & 0 \\ 0 & 1 & 0 & & & & & & 0 & 1 & 0 \\ 1 & 0 & 0 & & & & & & 0 & 0 & 1 \end{bmatrix}, \text{ is also a unit.}$$

The general even case, is simpler. Just take

$$U_{2k}=\begin{bmatrix} -1 & 0 & & & & & 0 & 1 \\ 0 & -1 & & & & & 1 & 0 \\ & & \ddots & & & \vdots & & \\ & & & -1 & 1 & & & \\ & & & 1 & 0 & & & \\ & & \vdots & & & \ddots & & \\ 0 & 1 & & & & & 0 & 0 \\ 1 & 0 & & & & & 0 & 0 \end{bmatrix}, \text{ for which}$$

$$U_{2k}+I_{2k}=\begin{bmatrix} 0 & 0 & & & & & 0 & 1 \\ 0 & 0 & & & & & 1 & 0 \\ & & \ddots & & & \vdots & & \\ & & & 0 & 1 & & & \\ & & & 1 & 1 & & & \\ & & \vdots & & & \ddots & & \\ 0 & 1 & & & & & 1 & 0 \\ 1 & 0 & & & & & 0 & 1 \end{bmatrix}$$ □

**Corollary 3.3.** *No matrix ring over a ring with identity is UU.*

*Proof.* The units $U+I_n$ in the previous proposition are not unipotent. □

**Alternative Proof**: It is clear that any ring in which identity is a sum of two units is not UU. Also it is well known that identity in a matrix ring $\mathcal{M}_n(R)$ with $n > 1$ is a sum of two units. For example see Lemma 1 of [5], where it is shown that every diagonal matrix in $\mathcal{M}_n(R)$ with $n > 1$ is a sum of two units. Thus $\mathcal{M}_n(R)$ with $n > 1$ is not UU.

## 4. UU GROUPS

To simplify wording, as customarily, Abelian groups with UU endomorphism ring, will be called UU groups (here "group" means "Abelian group"). Since $2$ (or $-2$) is nilpotent in any UU ring, multiplication by 2 must be a nilpotent endomorphism for any UU group. Hence such a group is bounded by a power of 2, and so, *only bounded direct sums of cyclic* $2$*-groups can be UU groups*.

As mentioned above, direct sums of UU groups need not be UU.

**Proposition 4.8.** *Only groups of the form* $\mathbf{Z}(2^{k_1}) \oplus \mathbf{Z}(2^{k_1}) \oplus ... \oplus \mathbf{Z}(2^{k_n})$ *with different* $k_i$*'s can be UU groups (finitely many because bounded).*

*Proof.* Since $\mathrm{End}(\mathbf{Z}(2^k) \oplus \mathbf{Z}(2^k)) \cong \mathcal{M}_2(\mathbf{Z}(2^k))$ is not UU, we just have to use Proposition 2.5. □

Before stating our main result, recall the following ingredients
**1**: [6] For a $p$-group $G$, $J(\mathrm{End}(G))$ is nilpotent if and only if the group $G$ is bounded.
**2**: [6] For a $p$-group $G$, $J(\mathrm{End}(G)) = 0$ if and only if $G$ is elementary.
**3**: [6] If $G$ is a torsion complete $p$-group then

$$\mathrm{End}(G)/J(\mathrm{End}(G)) \cong \prod_{n \geq 0} L_n,$$

where $L_n$ is the ring of linear transformations of an $F_p$-space of dimension $f_n(G)$. Here by $f_n(G)$ we denote the $n$th Ulm-Kaplansky invariant of $G$.

**4**: Any finite abelian $p$-group $G$ is isomorphic to $\mathbf{Z}(p^{\lambda_1}) \times \mathbf{Z}(p^{\lambda_2}) \times ... \times \mathbf{Z}(p^{\lambda_n})$ for some choice of integers $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq 0$, where $\mathbf{Z}(m)$ denotes the cyclic group of order $m$. The formula (see [8], Chapter II, Theorem 1.6) for the order of $\mathrm{Aut}(G)$, in terms of $\lambda_1$, $\lambda_2$, ..., $\lambda_n$, is

$$a_\lambda(p) = p^{|\lambda| + 2n(\lambda)} \prod_{i=1}^{n} \varphi_{m_i(\lambda)}(p^{-1})$$

where $\varphi_m(t) = (1-t)(1-t^2)...(1-t^m)$, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$, $|\lambda| = \sum_{i=1}^n \lambda_i$, $n(\lambda) = \sum_{i=1}^n (i-1)\lambda_i$ and the multiplicities $m_i(\lambda) = \mathrm{card}\{j : \lambda_j = \lambda_i\}$.

**Lemma 4.1.** *Let $G$ be a finite Abelian* $2$*-group of rank $n$. Then there is an isomorphism*
$\mathrm{End}(G)/2\mathrm{End}(G) \cong \mathcal{M}_n(\mathbb{F}_2)$.

*Proof.* There is a direct decomposition $G = C_1 \oplus \cdots \oplus C_n$, where all $C_i$ are cyclic subgroups of $G$. Thus $\mathrm{End}(G)$ is isomorphic to the matrix ring $(\mathrm{Hom}(C_i, C_j))_{1 \leq i,j \leq n}$ (see, e.g., [4]). Moreover, all groups $\mathrm{Hom}(C_i, C_j)$ are cyclic 2-groups. Thus, it is easy to see $\mathrm{End}(G)/2\mathrm{End}(G) \cong \mathcal{M}_n(\mathbb{F}_2)$, as stated. □

Now, here is the main result

**Theorem 4.1.** *An Abelian group $G$ has UU endomorphism ring if and only if it has the form* $G = \mathbf{Z}(2^{k_1}) \oplus \mathbf{Z}(2^{k_1}) \oplus ... \oplus \mathbf{Z}(2^{k_n})$ *with different* $k_i$*'s.*

*Proof.* Using the above Proposition and Proposition 2.1, we only need to show that finite direct sums of the given form are UU groups. Clearly, we may suppose $k_1 > k_2 > ... > k_n$.

Recall (see Introduction) that a finite ring $R$ is UU if and only if $|N(R)| = |U(R)|$. Therefore, in order to prove the statement it suffices to count nilpotent elements and units in $\mathrm{End}(G)$.

As for units, we show that $|\mathrm{Aut}(G)| = \frac{1}{2^n}\,|\mathrm{End}(G)|$. From **4**, in our special case ($p = 2$, nonisomorphic cyclic summands, $\lambda = (k_1, k_2, ..., k_n)$) we obtain
$|\mathrm{Aut}(G)| = 2^{k_1+k_2+...+k_n+2(k_2+2k_3+...(n-1)k_n)} \times 2^{-n}$, that is
$|\mathrm{Aut}(G)| = 2^{k_1+3k_2+5k_3+...+(2n-1)k_n-n}$.

Indeed, all $m_j(\lambda)$ are zero excepting $m_{k_i}(\lambda) = 1$ for $1 \leq i \leq n$.

The cardinal $|\mathrm{End}(G)| = 2^s$ where $s = k_1+3k_2+...+(2n-1)k_n$ is obtained as the cardinal of the matrix ring $(\mathrm{Hom}(\mathbf{Z}(2^{k_i}), \mathbf{Z}(2^{k_j}))_{1\leq i,j\leq n}$, observing that $\mathrm{Hom}(\mathbf{Z}(2^{k_i}), \mathbf{Z}(2^{k_j}) \cong \mathbf{Z}(2^{k_j})[2^{k_i}] \cong \left\{ \begin{array}{lll} \mathbf{Z}(2^{k_j}) & \text{if} & i \geq j \\ \mathbf{Z}(2^{k_i}) & \text{if} & i < j \end{array} \right.$ (see [4]).

Finally, according to **1**, 2-groups of the above type have nilpotent (and so, also nil) radical of the endomorphism ring (i.e., $J(R) = N(R)$), so it suffices to count the cardinal of the radical. According to **2**, these have nonzero radical.

Using **3**, for such 2-groups, all Ulm-Kaplansky are zero excepting

$f_{k_1-1}(G) = f_{k_2-1}(G) = ... = f_{k_n-1}(G) = 1$. Hence $\mathrm{End}(G)/J(\mathrm{End}(G)) \cong \prod_{s=1}^{n} F_2$ and so $2^n\,|J(\mathrm{End}(G))| = |\mathrm{End}(G)|$. Thus
$|N(\mathrm{End}(G))| = |J(\mathrm{End}(G))| = \frac{1}{2^n}\,|\mathrm{End}(G)| = |\mathrm{Aut}(G)|$, as desired. □

**Alternative Proof**: As observed in Prop. 4.8, it is clear that if $G$ is an abelian UU group, then $G = \mathbf{Z}_{2^{n_1}} \oplus \mathbf{Z}_{2^{n_2}} \oplus ... \oplus \mathbf{Z}_{2^{n_k}}$ with $n_1 < n_2 < ... < n_k$. Here is an alternative proof for "every group of the above form is UU". As each $\mathbf{Z}_{2^{n_i}}$ has local endomorphism ring and no two $\mathbf{Z}_{2^{n_i}}$ and $\mathbf{Z}_{2^{n_j}}$ are isomorphic, it is well known that $\mathrm{End}(G) = J(\mathrm{End}(G)) = \prod_{i=1}^{k} \mathrm{End}(\mathbf{Z}_{2^{n_i}}) = J(\mathrm{End}(\mathbf{Z}_{2^{n_i}})) = \mathbf{Z}_2^k$; which clearly is UU. Thus $\mathrm{End}(G)$ is also UU as $J(\mathrm{End}(G))$ is nilpotent.

**Remark 4.4.** Although this proof is shorter, we have also included the first because of the byproducts on $\mathrm{End}(G)$ and $\mathrm{Aut}(G)$, emphasized in it.

**Example 4.1.** Take $G = \mathbf{Z}(2) \oplus \mathbf{Z}(4)$. Using the Pierce decomposition $\mathrm{End}(\mathbf{Z}(2) \oplus \mathbf{Z}(4)) \cong \begin{bmatrix} \mathrm{End}\,(\mathbf{Z}(2)) & \mathrm{Hom}(\mathbf{Z}(4), \mathbf{Z}(2)) \\ \mathrm{Hom}(\mathbf{Z}(2), \mathbf{Z}(4)) & \mathrm{End}\,(\mathbf{Z}(4)) \end{bmatrix}$, we make the following notations:
$\mathrm{End}\,(\mathbf{Z}(2)) = \{0_{\mathbf{Z}(2)}, 1_{\mathbf{Z}(2)}\}$, $\mathrm{End}\,(\mathbf{Z}(4)) = \{0_{\mathbf{Z}(4)}, 1_{\mathbf{Z}(4)}, \alpha, \beta\}$ with
$\alpha = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 0 & 2 \end{pmatrix}$, $\beta = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 2 & 1 \end{pmatrix}$, $\mathrm{Hom}(\mathbf{Z}(4), \mathbf{Z}(2)) = \{f_0, f_1\}$ and
$\mathrm{Hom}(\mathbf{Z}(2), \mathbf{Z}(4)) = \{g_0, g_1\}$ with zero morphisms $f_0, g_0$ and $f_1(1) = 1$, $g_1(1) = 2$.

Summarizing, among the 32 matrices, there are 10 idempotent and 8 nilpotent matrices. Therefore, we have 8 unipotent matrices ($\begin{bmatrix} 1 & f_0 \\ g_0 & 1 \end{bmatrix} + N$), and another 8 matrices (indeed, $\begin{bmatrix} 1 & f_0 \\ g_0 & 1 \end{bmatrix}$ is the only unit *and* idempotent, and $\begin{bmatrix} 0 & f_0 \\ g_0 & 0 \end{bmatrix}$ is nilpotent *and* idempotent). These have idempotent square, and so, are not units. Hence all units are unipotent and the endomorphism ring is UU.

Moreover, for this ring $R \setminus U(R) \subseteq N(R)$ fails.

**Remark 4.5.** 1) If $R$ is UU then $R/I$ may not be UU.

Indeed, the Abelian 2-group $G = \mathbf{Z}(2^{k_1}) \oplus \mathbf{Z}(2^{k_1}) \oplus ... \oplus \mathbf{Z}(2^{k_n})$ with different $k_i$'s, has a UU endomorphism ring, but (by Corollary 3.3 and Lemma 4.1) $\mathrm{End}(G)/2\mathrm{End}(G) \cong \mathcal{M}_n(\mathbb{F}_2)$ is not UU. Notice that here $2\mathrm{End}(G)$ is a nil ideal.

2) In [1], it was proved that *a finite rank Abelian group G is nil-clean if and only if G is a finite 2-group*.

Therefore, UU endomorphism rings of Abelian groups are nil-clean. However, this is not true for general UU rings. Indeed, this follows from Corollary 2.2, since polynomial rings (over nonzero rings) are not (even) clean.

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