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UN-rings

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A nonzero ring is called a *UN*-ring if every nonunit is a product of a unit and a nilpotent element. We show that all simple Artinian rings are *UN*-rings and that the *UN*-rings whose identity is a sum of two units (e.g. if 2 is a unit), form a proper class of 2-good rings (in the sense of P. Vámos). Thus, any noninvertible matrix over a division ring is the product of an invertible matrix and a nilpotent matrix.

Keywords: Units; nilpotents; matrix rings; simple Artinian rings; 2-good rings; *UN*-rings.

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1. Introduction

A ring is called *clean* if every element is a sum of a unit and an idempotent. It is natural to ask which are the multiplicatively clean rings, that is, rings in which every element is a product of a unit and an idempotent. It turns out that this is a well-known class of rings: The *unit-regular* rings (a special class of von Neumann *regular* rings).

A ring is called *nil-clean* if every element is a sum of an idempotent and a nilpotent element. One can think of a multiplicatively analogue for nil-clean rings, that is, rings in which every element is a product of an idempotent and a nilpotent element. If we restrict ourselves to rings with identity, this class has no interest: indeed, it is readily seen that such a ring cannot have identity (unless zero).

Recently, rings in which every nonzero element is a sum of a unit and a nilpotent element were studied, under the name of *fine* rings (see [1]). It turns out that this is a new (proper) class of simple rings which (properly) contains the simple Artinian rings (i.e. full matrix rings over division rings).

Prompted by the above considerations, we introduce and investigate a new class of rings, which we call *UN*-rings. For any ring R, $U(R)$ denotes the group of units and $N(R)$ the set of all the nilpotent elements.

Definition. An element a of a ring with identity R is called a *UN*-element if there exist $u \in U(R)$ and $t \in N(R)$ such that $a = ut$. Denote $UN(R) = U(R)N(R)$ the set of all the *UN*-elements of R. Clearly every nilpotent element is *UN* but no unit is *UN* (because $U(R) \cap N(R) = \varnothing$ unless $R = 0$). Therefore we define a ring to be *UN*-ring if $R - U(R) = UN(R)$, that is, every nonunit is a *UN*-element.

For instance, any division ring is a *UN*-ring, and the converse holds for reduced rings.

Notice that *UN*-rings can also be described as rings R in which every nonunit is equivalent to a nilpotent element. (Two elements $a, b \in R$ are said to be *equivalent* if $b = uav$ for some $u, v \in U(R)$.)

In Sec. 2, we gather some basic properties of *UN*-elements and *UN*-rings and we show that such rings have the property $R - U(R) = U(R) + U(R)$, that is, if the identity is a sum of two units (e.g. if 2 is a unit), these are 2-good in the sense of Vámos (see [7]), a class which includes also a large part of the unit-regular rings (those for which 2 is a unit; see [2]). In Sec. 3 some partial results concerning the passage to corners of the *UN* property are presented and, in Sec. 4 results on *UN*-matrix rings are obtained. Among others, it is proved that simple Artinian rings are *UN*-rings. More, examples comparing fine elements and *UN*-elements are given in $\mathcal{M}_2(\mathbf{Z})$ and we show that square matrices over commutative elementary divisor rings are *UN* iff these have nilpotent determinant. To stimulate future work, open questions are stated all over when justified.

Definitions, notations and well-known results of Ring Theory follow those in [5].

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2. Simple Facts

As this was done for clean, nil-clean and fine elements, it is natural to define a nonunit element $a \in R$ to be *strongly UN* if there exists a *UN*-decomposition $a = ut$ such that $ut = tu$. If $R \neq 0$ and all nonunits are strongly *UN*, we say that R is a *strongly UN-ring*. Moreover, a nonzero ring R is *uniquely UN* if every nonunit has a unique *UN*-decomposition. Moreover, an element $a \in R$ is a *NU-element* if there is a unit $u \in U(R)$ and a nilpotent $t \in N(R)$ such that $a = tu$ and R is a *NU-ring* if all nonunits are *NU*. However, in the following result we show that these natural definitions do not give anything new.

Proposition 1. *For any ring R, the following statements hold.*

(1) *R is uniquely UN iff R is a division ring.*
(2) *R is strongly UN iff it is local with nil maximal ideal.*
(3) *$a \in R$ is a UN-element iff a is a NU-element.*

Proof. (1) It is easy to show that uniquely *UN*-rings are reduced and so division rings. Indeed, for any nilpotent $t \in N(R)$, $t = 1t = (1+t)[(1+t)^{-1}t]$ (t and $(1+t)^{-1}$ commute) and so $1 = 1 + t$ implies $t = 0$.

(2) Suppose R is strongly *UN* and let s be a nonunit. Then $s = ut$ with $u \in U(R)$, $t \in N(R)$ and so $s \in N(R)$ since $ut = tu$. Hence $R = U(R) \cup N(R)$ and so R is local with nil maximal ideal. Conversely, if R is local with nil maximal ideal, nonunits are nilpotent and so trivially strongly *UN*.

(3) Let $u \in U(R)$ and $t \in N(R)$. Since $tu = u(u^{-1}tu)$ and $u^{-1}tu$ is nilpotent together with t, the left-right symmetric definition is equivalent. □

In the next result $Z(R)$ denotes the center of R and $J(R)$ its Jacobson radical.

Proposition 2.

(1) *For any nonzero ring* $UN(R) \cap Z(R) = N(Z(R))$.
(2) *If* $u, v \in U(R)$ *then* $uUN(R)v = UN(R)$ *and* $vUN(R) = UN(R)$.
(3) *A commutative ring is UN iff it is local with nil maximal ideal.*
(4) *A local ring is UN iff* $UN(R) = J(R)$.

Example. Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. Then $\mathbf{Z}(n)$ is a *UN*-ring iff $k = 1$.

Notice that according to (2), when checking elements to be *UN*, we can use conjugates, but (since *NU*-elements are also *UN*) we can also use left or right multiplications with units.

As for nonexamples, *domains* (*even commutative*) *which are not division rings are not UN-rings*, because zero is the only nilpotent, and there are nonzero elements which are not units.

Example. In $\mathbf{Z}$, 0 is the only *UN*-element.

Proposition 3.

(1) *Factor rings of UN-rings are UN, but the converse fails.*
(2) *Direct products of UN-rings are not UN.*
(3) *Triangular rings are not UN.*
(4) *Polynomial rings over commutative rings are not UN.*
(5) *Split extensions of UN-rings may not be UN* (*i.e. if* R *is a ring,* $e \in R$ *is an idempotent and* eRe *and* $(1-e)R(1-e)$ *are both UN-rings then* R *may not be UN*).

Proof. (1) Indeed, $\mathbf{Z}/4\mathbf{Z}$ is *UN* but $\mathbf{Z}$ is not. Notice that for an ideal I in a ring R, $a \notin U(R)$ does not imply $a + I \notin U(R/I)$.

(2) Let $R \times S$ be a direct product with $u \in U(R)$ and $s \notin U(S)$. Then $(u, s) \notin U(R \times S)$ so if it would have a *UN*-decomposition, u should have a *UN*-decomposition in R, a contradiction.

(3) Follows from (2). Example: $\mathcal{T}_2(\mathbf{F}_2)$, the ring of all upper triangular 2×2 matrices over $\mathbf{F}_2$.

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(4) Since this ring is commutative $UN(R) = N(R)$. However $X \in R[X]$ is not a unit nor nilpotent.

(5) In the ring $R = \mathbf{Z}(6)$, which is not *UN*, take the idempotent $e = \widehat{3}$. The corners $eRe \cong \mathbf{Z}(2)$ and $(1-e)R(1-e) \cong \mathbf{Z}(3)$ are both *UN*. □

The following result shows that to some extent *UN*-rings are not so far from fine rings (however, if $\Phi(R)$ denotes the fine elements of R, then $U(R) \subseteq \Phi(R)$ but $U(R) \cap UN(R) = \varnothing$ for nonzero rings). According to Vámos (see [7]) an element in a ring is *2-good* if it is a sum of two units.

Proposition 4. *Any UN-element is* 2*-good, and so* $R - U(R) = U(R) + U(R)$ *holds for any UN-ring.*

Proof. Indeed, if $a = ut$, then $a = u(1+t) - u \in U(R) + U(R)$. □

Conversely, a 2-good element $u+v$ is *UN* if at least one of $-u^{-1}v$, $-vu^{-1}$, $-uv^{-1}$ or $-v^{-1}u$ is unipotent.

Corollary 5. *UN-rings whose identity is a sum of two units are* 2*-good.*

Proof. Indeed, by the previous proposition, all elements are 2-good iff the units are 2-good iff the identity 1 is 2-good. □

Therefore, *UN*-rings are 2-good whenever 2 is a unit, but $\mathbf{F}_2$ is *UN* and not 2-good.

Note that (excepting $\mathbf{F}_2$) fine rings are also 2-good: indeed, for a fine ring it is not hard to check that (1) $R = \{1\} \cup (U(R) + U(R))$, and then (2) R is 2-good iff $|R| \neq 2$ ([1, Theorem 2.8]).

Question. Refine the inclusion $\{$*UN*-rings with 2-good identity$\} \subset \{$2-good rings$\}$, that is, find classes $\mathcal{C}$ of rings such that $\{$*UN*-rings with 2-good identity$\} \subset \mathcal{C} \subset \{$2-good rings$\}$.

Since quasiregular elements are 2-good (r is said to be *quasiregular*, if $1 - r$ is a unit in R and so $r = 1 - (1 - r)$) a comparison with *UN*-elements is in order. A quasiregular element may not be *UN* since units can be quasiregular:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = I_2 - \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}.$$

Conversely, *UN*-elements may not be quasiregular:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

is *UN* but not quasiregular since

$$I_2 - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

is not a unit.

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3. Corners of *UN*-Rings

This section is devoted to the consideration of the important question *whether the notion of UN-ring is Morita-invariant*, that is, whether the property of being a *UN*-ring is preserved by the Morita equivalence of rings. By a standard result in the literature on Morita equivalences, we should first answer the following:

Question. If R is a *UN*-ring and $e \in R$ is a full idempotent (that is, an idempotent such that $ReR = R$), is the corner ring eRe necessarily a *UN*-ring?

For rings generated by units, the similar question was already asked in [6], and answered (see [3]) in the negative by choosing T the ring $F[x]$ of all polynomials over a field (which is not generated by units), $R = \mathcal{M}_2(T)$, and

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $eRe \cong T$ even though R is 2-good.

One may begin by asking more generally how the *UN*-elements of eRe are related to those of R. Even without assuming the idempotent e to be full (here $\overline{e}$ denotes the complementary idempotent of e), the well-known equalities $N(eRe) = (eRe) \cap N(R)$ and $U(eRe) = (eRe) \cap (\overline{e} + U(R))$ do not permit to compare $UN(eRe)$ with $UN(R)$.

Less than giving an answer to the above question, one might wonder if *some* relation between $UN(eRe)$ and $UN(R)$ may be better suited for getting a positive answer. For instance, if it is true that $eRe \cap UN(R) \subseteq UN(eRe)$ (for any full idempotent $e \in R$), then certainly this question would have a "yes" answer. Unfortunately, this inclusion relation *does not* hold in general, as the following examples will show.

Example 1. Taking $R = \mathcal{M}_3(\mathbf{Z})$ and $e \in R$ to be the full idempotent $\operatorname{diag}(1, 1, 0)$, we identify $S := eRe$ with $\mathcal{M}_2(\mathbf{Z})$ (which corresponds to the "2×2 northwest corner" of $\mathcal{M}_3(\mathbf{Z})$). Let $A = I_2$ which, as unit, is not a *UN*-element of S. As an element of S, A is identified with the 3×3 matrix $\operatorname{diag}(A, 0) \in R$. This matrix turns out to be in *UN*(R) since it has the following *UN*-decomposition:

$$\operatorname{diag}(A, 0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = UT = \begin{bmatrix} a & 1 & 0 \\ b & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

with $T^3 = 0_3$ and $\det(U) = 1$ (for any $a, b \in \mathbf{Z}$).

Example 2. We will show here that the inclusion relation $eRe \cap UN(R) \subseteq UN(eRe)$ may fail even in the case where R is a 2×2 matrix ring over some ring S and $e = \operatorname{diag}(1, 0)$ (which is a full idempotent of R that is similar to its complementary idempotent). To get such an example, let $S = \mathcal{M}_2(\mathbf{Z})$ as in Example 1. For this choice of S, $R := \mathcal{M}_2(S)$ may be identified with $\mathcal{M}_4(\mathbf{Z})$, and S may

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then be identified with eRe. Taking again the identity matrix as in Example 1, we can check again that

$$\operatorname{diag}(A,0,0)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}=UT=\begin{bmatrix}1&1&1&-1\\0&-1&1&0\\0&0&1&-1\\1&0&1&-1\end{bmatrix}\begin{bmatrix}0&0&0&0\\1&0&0&0\\1&1&0&0\\1&1&0&0\end{bmatrix}$$

with $T^4=0_4$ and $\det(U)=1$.

Note that, alternatively, we could conclude that these two examples are *UN*, by using Lemma 6 (next section).

However, the fact that in general $eRe\cap UN(R)\nsubseteq UN(eRe)$ for full idempotents $e\in R$, should not entirely dash our hopes for a positive answer. This is because, in working with this question, we are under the strong assumption that *all* (not just some) elements of the ring R are *UN*.

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4. *UN*-Matrix Rings

First observe that fine element and *UN*-element are independent notions. Indeed, since $\left[\begin{smallmatrix}a&b\\0&0\end{smallmatrix}\right]$ is fine in $\mathcal{M}_2(\mathbf{Z})$ iff $b\equiv\pm1 \pmod a$ (see [1, Corollary 5.4]), for instance $\left[\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right]$ is *UN* but not fine (nor nilpotent). In the reverse direction,

$$\begin{bmatrix}1&2\\3&0\end{bmatrix}=\begin{bmatrix}-1&3\\-1&2\end{bmatrix}+\begin{bmatrix}2&-1\\4&-2\end{bmatrix}$$

is fine but not *UN* (since $\det=-6\neq0$). However

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}2&1\\-1&-1\end{bmatrix}+\begin{bmatrix}-1&-1\\1&1\end{bmatrix}$$

is (nonzero) *UN and* fine.

Next, let $\pi\in S_n$ be a permutation. We associate to π a so-called *permutation matrix* P_π which is obtained from the identity matrix I_n by the corresponding permutation of rows.

In obtaining our results on *UN*-matrix rings we use the following key

Lemma 6. *Let R be a ring, $n\geq 2$, $d_1,d_2,\ldots,d_n\in R$ and P the $n\times n$ permutation matrix corresponding to the n-cycle* $(1,2,\ldots,n)$. *If Δ is the diagonal matrix* $\operatorname{diag}(d_1,d_2,\ldots,d_n)$ *then* $(P\Delta)^n=\operatorname{diag}(d'_1,d'_2,\ldots,d'_n)$ *where* $d'_i=d_id_{i+1}\cdots d_nd_1\cdots d_{i-1}$, $1\leq i\leq n$. *Therefore, if one of the d_j's is nilpotent and commutes with the other d_i's, then $P\Delta$ is nilpotent.*

Proof. It is readily seen that the nonzero positions in $(P\Delta)^k$ for any $1\leq k\leq n$ are the same with the nonzero positions in P^k (and so the only diagonal $(P\Delta)^k$

is $(P\Delta)^n$). The nonzero entries can be found as follows. Denote by δ the n-vector $(d_1, d_2, \ldots, d_n)$, by π the cycle $(1, 2, \ldots, n)$ and by $\pi^{-1}(\delta)$ the vector obtained by applying the permutation π^{-1} to the entries of δ. Then the nonzero entries in $(P\Delta)^k$, written in the order of columns are given by the componentwise product $\pi^{-(k-1)}(\delta) \cdots \pi^{-2}(\delta)\pi^{-1}(\delta)\delta$. Hence the first claim. The second claim is straightforward. □

Recall that a ring is an *elementary divisor* ring (see [4]) if every square matrix can be diagonalized. Elementary divisor rings form a subclass of Bézout rings (rings in which every finitely generated left or right ideal is principal) and include PIDs, left PIDs which are Bézout (in particular division rings), valuation rings, the ring of entire functions, etc.

Corollary 7. *Let D be a division ring, $n \geq 2$ and $\mathcal{M}_n(D)$ the ring of $n \times n$ matrices over D. Then $\mathcal{M}_n(D)$ is a UN-ring.*

Proof. We just have to notice that D is an elementary divisor ring, and so every matrix is equivalent to a diagonal one, with (at least) the last diagonal entry zero if the matrix is not invertible. Then we apply the previous Lemma. □

Corollary 8. *Simple Artinian rings are UN-rings.*

Remark. *Not all nonzero UN-rings are simple Artinian.*

Indeed, take a chain of simple Artinian rings $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$ which share the same identity and the union $R = \bigcup_{i\geq 0} R_i$. This is also a simple ring (see [5, p. 40)] but is not Artinian. However, by the previous Corollary, each R_i is a *UN*-ring and then so is R.

Further, since matrix rings over 2-good rings are 2-good (see [7, Theorem 11]), it is natural to ask whether this statement has a valid analogue for *UN*-rings. Since troubles occur already for 2×2 matrices, we were concentrating on the following:

Question. Are 2×2 matrices over *UN*-rings, also *UN*?

Unfortunately we were not able to answer in the affirmative in all cases. We were able just to prove the following result.

To simplify our statement, two nilpotents t and t' are called *independent* if none is a ring multiple of the other (equivalently, $t \notin t'R \cup Rt'$ and $t' \notin tR \cup Rt$).

Proposition 9. *All noninvertible 2×2 matrices over a UN-ring are UN, with one possible exception: 2×2 matrices of the form $\left[\begin{smallmatrix} t & vt' \\ wt'' & t''' \end{smallmatrix}\right]$ with $v, w \in U(R)$ and nonzero nilpotent entries t, t', t'', t''', independent on rows and on columns.*

Since this is only a partial result, we have decided not to include its proof (which is three pages long!).

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We suspect that the question above has a negative answer, but, despite the detailed analysis made in the proof of the previous proposition, we were not able to find a counterexample, not even for matrices with four nonzero nilpotent entries.

It is known (see [3]) that matrix rings over elementary divisor rings are 2-good. This justifies the following:

Question. Are matrix rings over elementary divisor *UN*-rings also *UN*-rings?

Using the consequence (conjugates, unit multiples) of Proposition 2(2), it suffices to find *UN*-decompositions for all diagonal matrices. Since we deal only with noninvertible matrices, not all entries on the diagonal are units. Since the case with no units on diagonal is straightforward, the problem reduces to diagonal matrices with 1's and nilpotents entries only, which can be brought to the block form $\left[\begin{smallmatrix} I_n & 0 \\ 0 & T \end{smallmatrix}\right]$. If

$$T = \operatorname{diag}(t_1, \ldots, t_m) \text{ then } \begin{bmatrix} I_n & 0 \\ 0 & T \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ I_m & 0 \end{bmatrix} \begin{bmatrix} 0 & T \\ I_n & 0 \end{bmatrix}$$

is a decomposition, but generally the right matrix in RHS is not nilpotent. However, over commutative rings we can do better as the following result shows.

Theorem 10. *Let R be a commutative elementary divisor ring and A a square matrix over R. Then A is a UN-matrix iff* $\det(A)$ *is nilpotent.*

Proof. Since the determinant of a *UN*-matrix with entries in a commutative ring R is nilpotent, only the converse needs justification. But since the product of the diagonal elements in the diagonalization of A is an associate (unit multiple) of the determinant of A, the result follows from Lemma 6. □

Corollary 11. *Matrix rings over commutative elementary divisor UN-rings are also UN-rings.*

Proof. Since we noticed that a commutative ring is *UN* iff it is local with nil maximal ideal, it follows that not invertible matrices have nilpotent determinant and the Theorem applies. □

Corollary 12. *All idempotents $\neq I_2$ in $\mathcal{M}_2(\mathbf{Z})$ are UN-matrices.*

The last corollary also suggests the following:

Question. In which rings are all the non-identity idempotents *UN*-elements?

Note that if an idempotent has a *UN*-decomposition, the corresponding nilpotent is unit-regular.

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References

[1] G. Călugăreanu and T. Y. Lam, Fine rings: A new class of simple rings, *J. Algebra and Its Appp.* **15**(9) (2015) (18 pages).
[2] G. Ehrlich, Unit-regular rings, *Portugal. Math.* **27** (1968) 209–212.
[3] M. Henriksen, Two classes of rings generated by their units, *J. Algebra* **31** (1974) 182–193.
[4] I. Kaplansky, Elementary divisors and modules, *Trans. Amer. Math. Soc.* **66** (1949) 464–491.
[5] T. Y. Lam, *A First Course in Noncommutative Ring Theory*, 2nd edn. Graduate Texts in Mathematics, Vol. 131 (Springer-Verlag, Berlin, 2001).
[6] R. Raphael, Rings which are generated by units, *J. Algebra* **28** (1974) 199–205.
[7] P. Vámos, 2-Good rings, *Quart. J. Math.* **56** (2005) 417–430.

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