Strongly regular rings have stable range one: a simple (independent) unitizer

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A ring R is called *strongly* (von Neumann) *regular* if for every a in R, there is some x in R with $a = a^2x$.

Recall that strongly regular elements in any reduced ring are regular. More precisely, if $a = a^2x$ then $(a - axa)^2 = 0$ and so $a = axa$. Consequently, ax is an idempotent and we show that the complementary idempotent $1 - ax$ is a unitizer for a.

Also recall that strongly regular rings are reduced and so Abelian and so Dedekind finite.

In order to check $sr(a) = 1$, we show that for every $r \in R$, $a+(1-ax)(1-ra)$ is a unit of R. Indeed, this follows from the formula

$$a + (1 - ax)(1 - ra) = [1 - (1 - ax)(a + 1 - ax)r(ax)](a + 1 - ax) \in U(R)$$

since $(1 - ax)a = a(1 - ax) = 0$, $(1 - ax)(a + 1 - ax)r(ax)$ is zero-square and $a + 1 - ax$ is a unit (with right inverse $ax^2 + 1 - ax$).