Integral 2×2 matrices, similar to transposes

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1 Introduction

An $n \times n$ Jordan block is similar to its transpose by the $n \times n$ involution U with 1's along the *cross-diagonal* (lower left to upper right) and zero's elsewhere.

So using such involutions $U_1, \ldots, U_l$ for each Jordan block $J_1, \ldots, J_l$ and letting U be the direct sum matrix of these $U_1, \ldots, U_l$, it follows that every Jordan form (i.e. direct sum of Jordan blocks) of a matrix, is similar to its transpose.

For any base ring such that any matrix is similar to its Jordan form, it follows that each matrix is similar to its transpose.

It is well-known that this happens if the base ring is an *algebraic closed field*.

Using companion matrices instead of Jordan blocks, the algebraic closed hypothesis can be removed (see [**1**]). Moreover, it is proved that the invertible matrix which realizes the similarity can be chosen symmetric.

However, over a *commutative domain* (even PID, e.g. $\mathbb{Z}$) this may fail.

2 Characterization

Below, matrices which are similar to their transpose are described over the integers. Obviously, symmetric matrices may be discarded from our discussion. The characterization reduces to two Diophantine equations and a divisibility.

Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $A \neq A^T$ (i.e. A is not symmetric). Notice that both b, c cannot be zero and suppose $b \neq 0$ (otherwise we discuss the transpose A^T).

Proposition 1 *A nonsymmetric integral matrix* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *with* $b \neq 0$ *is similar to its transpose iff at least one of the Diophantine equations*

$$cx^2 + (d-a)xy - by^2 \pm b = 0$$

is solvable (over $\mathbb{Z}$*) and for at least one solution* (x_0, y_0)*,* b *divides* $cx_0 + (d-a)y_0$*.*

Proof. Denote $U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$. The equality $AU = UA^T$ amounts to the equalities $u_{12} = u_{21}$, $bu_{22} = cu_{11} + (d-a)u_{12}$. To these we must add $\det(U) = \pm 1$ for U to be invertible. Hence $u_{11}u_{22} - u_{12}u_{21} = \pm 1$ which, by multiplication withy $b \neq 0$, gives $u_{11}[cu_{11} + (d-a)u_{12}] - bu_{12}^2 = \pm b$, whence the Diophantine equations in the statement (for $u_{11} = x$, $u_{12} = y$). The divisibility condition follows from $bu_{22} = cu_{11} + (d-a)u_{12}$. ■

Remarks. 1) If $c = d - a$ we can chose $u_{22} = 0$, $u_{12} = u_{21} = 1$, $u_{11} = -1$ (i.e. $U = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$).

2) If b divides $d - a$ we can chose $u_{11} = 0$, $u_{12} = u_{21} = 1$ and $u_{22} = \frac{d-a}{b}$ (i.e. $U = \begin{bmatrix} 0 & 1 \\ 1 & \frac{d-a}{b} \end{bmatrix}$).

Therefore, in order to improve chances to find a matrix not similar to its transpose we should chose (this is the case below), $c \neq d - a$ and b not divisor of $d - a$.

Example. Take $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. We get two Diophantine equations: $3x^2 + 5xy - 2y^2 - 2 = 0$ which has no (integer) solutions, and, $x^2 + 5xy - 2y^2 + 2 = 0$ which has only the (obvious) solutions $(0, \pm 1)$. However, 2 does not divide ± 5, so A is **not** similar to its transpose.

This can be generalized to matrices $\begin{bmatrix} 1 & s \\ s+1 & s(s+1) \end{bmatrix}$.

In this case, the Diophantine are: $(s+1)x^2 + [s(s+1) - 1]xy - sy^2 \pm s = 0$.

For $-s$ *no* solutions and for $+s$, *only* the solutions $(0, \pm 1)$ [proof?]. Then $su_{22} = \pm[s(s+1) - 1]$, obviously has no integer solution.

References

[1] O. Taussky, H. Zassenhaus *On the similarity transformation between a matrix and its transpose*. Pacific Journal of Mathematics, **9** (1959), 893-896.