FROM REDUCED RINGS TO DEDEKIND FINITE RINGS

In [5], **Exercise 12.18**, the following implications are stated:

reduced $\Rightarrow$ symmetric $\Rightarrow$ reversible $\Rightarrow$ 2-primal.

There are another possible implications, one can add, instead of 2-primal:

reduced $\Rightarrow$ symmetric $\Rightarrow$ reversible $\Rightarrow$ semicommutative $\Rightarrow$ Abelian $\Rightarrow$ Dedekind finite.

Notice that all these implications are irreversible. For the first two, examples are given in T.Y. Lam *Exercises in Classical Ring Theory* (corresponding to the Exercise mentioned before).

For the next two examples are given in [3], and any matrix ring over a commutative ring is Dedekind finite but not Abelian.

Definitions. A ring R is *reduced* if it has no nonzero nilpotents, *symmetric* if for $a, b, c \in R$, $abc = 0 \Rightarrow bac = 0$, *reversible* (see [2]) if for all $a, b \in R$: $ab = 0 \Longrightarrow ba = 0$, *semicommutative* if for every $a \in R$, $r_R(a)$ is an ideal of R (equivalently, $l_R(a)$ is an ideal of R), *Abelian* if its idempotents are central, and is *Dedekind finite* if one-sided invertible elements are two-sided (i.e., for all $a, b \in R : ab = 1 \Longrightarrow ba = 1$).

In order to have all (direct) proofs in one place we supply these here and add some others.

Lemma 1. *(i) Reduced rings are reversible.*
(i') Reduced rings are symmetric.
(ii) Symmetric rings are reversible.
(iii) Reversible rings are semicommutative.
(iv) Semicommutative rings are Abelian.
(v) Abelian rings are Dedekind finite.

Proof. (i) If $ab = 0$ then $(ba)^2 = baba = 0$ and so $ba = 0$.

(i') Suppose $abc = 0$. We also repeatedly use (i) in the (not trivial) proof (attributed to Andrunakievič, Ryabukhin; see [1]). It goes like this: $(ab)c = 0 \Rightarrow cab = 0 \Rightarrow c(aba) = 0 \Rightarrow abac = 0 \Rightarrow (ba)(bac) = 0 \Rightarrow bacba = 0 \Rightarrow (bac)^2 = 0 \Rightarrow bac = 0$.

(ii) In $abc = 0 \Rightarrow bac = 0$, just take $c = 1$.

(iii) For any $a \in R$, $r_R(a)$ is clearly closed under addition and right multiplication. It only remains to show that $ax = 0 \Rightarrow abx = 0$ for any $b \in R$. By reversibility, $xa = 0$ and so $bxa = 0$. Again by reversibility $abx = 0$.

(iv) (Shin). If $e^2 = e$ then $e\overline{e} = \overline{e}e = 0$ means that $\overline{e} \in r_R(e)$ and $e \in r_R(\overline{e})$. Since these are ideals, $ea(1 - e) = 0$ and $(1 - ea)e = 0$. Hence $ea = eae = ae$.

(v) Suppose $ab = 1$. Then $(ba)^2 = ba$ is an idempotent, so central by hypothesis. Thus $b = (ba)b = b(ba) = b^2a$ and so $1 = ab = ab^2a$.

Finally, $ba = (ab^2a)ba = (ab^2)(ab)a = ab^2a = 1$. $\square$

Some other direct proofs.

Lemma 2. *(i) Reduced rings are reversible.*
(ii) Reduced rings are Abelian.
(ii) Reversible rings are Dedekind finite.

Proof. (i) If $ab = 0$, for some $a, b \in R$, then $(ba)^2 = b(ab)a = 0$ and thus $ba = 0$.

(ii) Let $e^2 = e \in R$ and $x \in R$. Computation shows that $(ex - exe)^2 = (xe - exe)^2 = 0$. Hence $ex = exe = xe$, i.e., e is central.

(iii) Suppose that $ab = 1$ for some $a, b \in R$. Then $(ba - 1)b = b(ab) - b = 0$ and thus $b(ba - 1) = 0$. So $b^2a = b$ and hence $ab^2a = ab = 1$. It follows that $ba = (ab^2a)ba = (ab^2)(ab)a = ab^2a = 1$. So R is Dedekind finite. □

From [2].

Lemma 3. *A semprime reversible ring is reduced.*

Proof. Suppose $t^2 = 0$. Then for every $x \in R$, $t^2x = 0$ and by reversibility, $txt = 0$. Hence $tRt = (0)$ and by semiprime ([5] **(10.9)**: An ideal I is *semiprime* iff $aRa \subseteq I$ implies $a \in I$), $t = 0$. □

Theorem 4. *In a reversible ring $N(R)$ is an ideal (i.e., is a so called NI ring).*

Proof. Let $x^r = y^s = 0$. Then $(x + y)^{r+s-1}$ is a sum of products of x and y, each product consisting of $r + s - 1$ factors. Each term has at least r factors x or at least s factors y. Since commutation is possible using reversibility, all factors vanish and $x + y \in N(R)$.

If $x^r = 0$ then $(bxc)^r$ has r factors x so vanish, for any $b, c \in R$. So $N(R)$ is a (two-sided) ideal. □

Examples of commutations:

1) Suppose $x^2 = y^3 = 0$. Then $(x + y)^4 = (x^2 + yx + xy + y^2)^2 =$
$= x^4 + yx^3 + xyx^2 + y^2x^2 + x^2yx + \underline{yxyx} + \underline{xy^2x} + y^3x + ...$

Now $x^2 = 0 \Rightarrow x^2y = 0 \overset{\text{rev}}{\Rightarrow} xyx = 0$ and $x^2y^2 = 0 \overset{\text{rev}}{\Rightarrow} xy^2x = 0$. And another eight products.

2) Suppose $x^3 = 0$. Then $cbx^3 = 0 \overset{\text{rev}}{\Rightarrow} xcbx^2 = 0 \Rightarrow bxcbx^2 = 0 \Rightarrow cbxcbx^2 = 0 \overset{\text{rev}}{\Rightarrow} xcbxcbx = 0$ and so $(bxc)^3 = b\underline{xcbxcbx}c = 0$.

References

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